



## Let's Study

- 4.1 Combined equation of a pair lines.
- 4.2 Homogeneous equation of degree two.
- 4.3 Angle between lines.
- 4.4 General second degree equation in  $x$  and  $y$ .

## 4.1 INTRODUCTION

We know that equation  $ax + by + c = 0$ , where  $a, b, c \in R$ , ( $a$  and  $b$  not zero simultaneously), represents a line in  $XY$  plane. We are familiar with different forms of equations of line. Now let's study two lines simultaneously. For this we need the concept of the combined equation of two lines.



## Let's learn.

## 4.1 Combined equation of a pair of lines :

An equation which represents two lines is called the combined equation of those two lines. Let  $u \equiv a_1x + b_1y + c_1$  and  $v \equiv a_2x + b_2y + c_2$ . Equation  $u = 0$  and  $v = 0$  represent lines. We know that equation  $u + kv = 0$ ,  $k \in R$  represents a family of lines. Let us interpret the equation  $uv = 0$ .

**Theorem 4.1:**

The equation  $uv = 0$  represents, the combined equation of lines  $u = 0$  and  $v = 0$

**Proof :** Consider the lines represented by  $u = 0$  and  $v = 0$

$$\therefore a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0.$$

Let  $P(x_1, y_1)$  be a point on the line  $u = 0$ .

$$\therefore (x_1, y_1) \text{ satisfy the equation } a_1x + b_1y + c_1 = 0$$

$$\therefore a_1x_1 + b_1y_1 + c_1 = 0$$

To show that  $(x_1, y_1)$  satisfy the equation  $uv = 0$ .

$$\begin{aligned} (a_1x_1 + b_1y_1 + c_1)(a_2x_1 + b_2y_1 + c_2) \\ = 0(a_2x_1 + b_2y_1 + c_2) \\ = 0 \end{aligned}$$

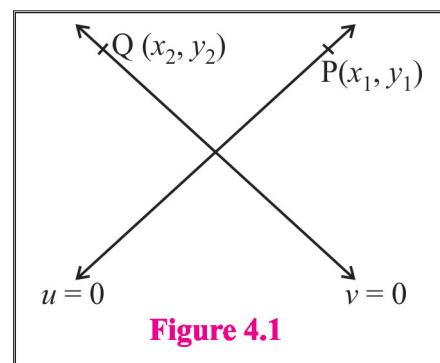
Therefore  $(x_1, y_1)$  satisfy the equation  $uv = 0$ .

This proves that every point on the line  $u = 0$  satisfy the equation  $uv = 0$ .

Similarly we can prove that every point on the line  $v = 0$  satisfies the equation  $uv = 0$ .

Now let  $R(x', y')$  be any point which satisfy the equation  $uv = 0$ .

$$\therefore (a_1x' + b_1y' + c_1)(a_2x' + b_2y' + c_2) = 0$$



$$\therefore (a_1 x' + b_1 y' + c_1) = 0 \text{ or } (a_2 x' + b_2 y' + c_2) = 0$$

Therefore  $R(x', y')$  lies on the line  $u = 0$  or  $v = 0$ .

Every points which satisfy the equation  $uv = 0$  lies on the line  $u = 0$  or the line  $v = 0$ .

Therefore equation  $uv = 0$  represents the combined equation of lines  $u = 0$  and  $v = 0$ .

### Remark :

- 1) The combined equation of a pair of lines is also called as the joint equation of a pair of lines.
- 2) Equations  $u = 0$  and  $v = 0$  are called separate equations of lines represented by  $uv = 0$ .



### Solved Examples :

**Ex. 1)** Find the combined equation of lines  $x + y - 2 = 0$  and  $2x - y + 2 = 0$

**Solution :** The combined equation of lines  $u = 0$  and  $v = 0$  is  $uv = 0$

$\therefore$  The combined equation of lines  $x + y - 2 = 0$  and  $2x - y + 2 = 0$  is

$$(x + y - 2)(2x - y + 2) = 0$$

$$\therefore x(2x - y + 2) + y(2x - y + 2) - 2(2x - y + 2) = 0$$

$$\therefore 2x^2 - xy + 2x + 2xy - y^2 + 2y - 4x + 2y - 4 = 0$$

$$\therefore 2x^2 + xy - y^2 - 2x + 4y - 4 = 0$$

**Ex. 2)** Find the combined equation of lines  $x - 2 = 0$  and  $y + 2 = 0$ .

**Solution :** The combined equation of lines  $u = 0$  and  $v = 0$  is  $uv = 0$ .

$\therefore$  The combined equation of lines  $x - 2 = 0$  and  $y + 2 = 0$  is

$$(x - 2)(y + 2) = 0$$

$$\therefore xy + 2x - 2y - 4 = 0$$

**Ex. 3)** Find the combined equation of lines  $x - 2y = 0$  and  $x + y = 0$ .

**Solution :** The combined equation of lines  $u = 0$  and  $v = 0$  is  $uv = 0$ .

$\therefore$  The combined equation of lines  $x - 2y = 0$  and  $x + y = 0$  is

$$(x - 2y)(x + y) = 0$$

$$\therefore x^2 - xy - 2y^2 = 0$$

**Ex. 4)** Find separate equation of lines represented by  $x^2 - y^2 + x + y = 0$ .

**Solution :** We factorize equation  $x^2 - y^2 + x + y = 0$  as

$$(x + y)(x - y) + (x + y) = 0$$

$$\therefore (x + y)(x - y + 1) = 0$$

Required separate equations are  $x + y = 0$  and  $x - y + 1 = 0$ .

## 4.2 Homogeneous equation of degree two:

### 4.2.1 Degree of a term:

**Definition:** The sum of the indices of all variables in a term is called the degree of the term.

For example, in the expression  $x^2 + 3xy - 2y^2 + 5x + 2$  the degree of the term  $x^2$  is two, the degree of the term  $3xy$  is two, the degree of the term  $-2y^2$  is two, the degree of  $5x$  is one. The degree of constant term 2 is zero. Degree of '0' is not defined.

### 4.2.2 Homogeneous Equation :

**Definition:** An equation in which the degree of every term is same, is called a homogeneous equation.

For example:  $x^2 + 3xy = 0$ ,  $7xy - 2y^2 = 0$ ,  $5x^2 + 3xy - 2y^2 = 0$  are homogeneous equations.

But  $3x^2 + 2xy + 2y^2 + 5x = 0$  is not a homogeneous equation.

Homogeneous equation of degree two in  $x$  and  $y$  has form  $ax^2 + 2hxy + by^2 = 0$ .

**Theorem 4.2 :**

The combined equation of a pair of lines passing through the origin is a homogeneous equation of degree two in  $x$  and  $y$ .

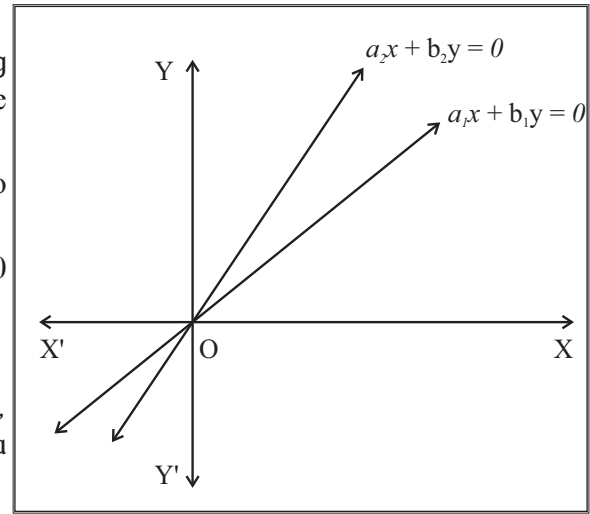
**Proof :** Let  $a_1x + b_1y = 0$  and  $a_2x + b_2y = 0$  be any two lines passing through the origin.

Their combined equation is  $(a_1x + b_1y)(a_2x + b_2y) = 0$

$$a_1a_2x^2 + a_1b_2xy + a_2b_1xy + b_1b_2y^2 = 0$$

$$(a_1a_2)x^2 + (a_1b_2 + a_2b_1)xy + (b_1b_2)y^2 = 0$$

In this if we put  $a_1a_2 = a$ ,  $a_1b_2 + a_2b_1 = 2h$ ,  $b_1b_2 = b$ , we get,  $ax^2 + 2hxy + by^2 = 0$ , which is a homogeneous equation of degree two in  $x$  and  $y$ .



**Figure 4.2**

**Ex.1)** Verify that the combined equation of lines  $2x + 3y = 0$  and  $x - 2y = 0$  is a homogeneous equation of degree two.

**Solution :**

The combined equation of lines  $u = 0$  and  $v = 0$  is  $uv = 0$ .

$\therefore$  The combined equation of lines  $2x + 3y = 0$  and  $x - 2y = 0$  is

$$(2x + 3y)(x - 2y) = 0$$

$$2x^2 - xy - 6y^2 = 0, \text{ which is a homogeneous equation of degree two.}$$

**Remark :**

The combined equation of a pair of lines passing through the origin is a homogeneous equation of degree two. But every homogeneous equation of degree two *need not* represent a pair of lines.

Equation  $x^2 + y^2 = 0$  is a homogeneous equation of degree two but it does not represent a pair of lines.

How to test whether given homogeneous equation of degree two represents a pair of lines or not?

Let's have a theorem.

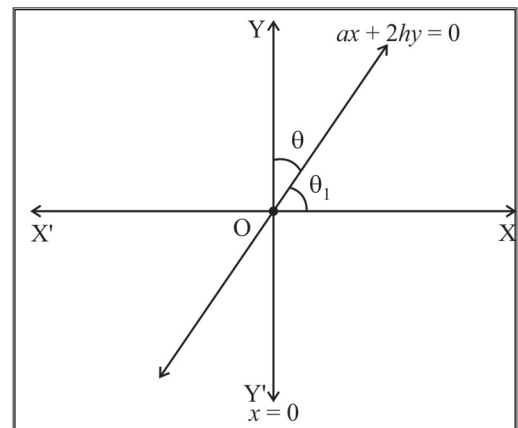
**Theorem 3 :** Homogeneous equation of degree two in  $x$  and  $y$ ,  $ax^2 + 2hxy + by^2 = 0$  represents a pair of lines passing through the origin if  $h^2 - ab \geq 0$ .

**Proof :** Consider the homogeneous equation of degree two in  $x$  and  $y$ ,  $ax^2 + 2hxy + by^2 = 0$  ..... (1)

Consider two cases  $b = 0$  and  $b \neq 0$ . These two cases are exhaustive.

**Case 1:** If  $b = 0$  then equation (1) becomes  $ax^2 + 2hxy = 0$

$\therefore x(ax + 2hy) = 0$ , which is the combined equation of lines



**Figure 4.3**

$$x = 0 \text{ and } ax + 2hy = 0.$$

We observe that these lines pass through the origin.

**Case 2:** If  $b \neq 0$  then we multiply equation (1) by  $b$ .

$$abx^2 + 2hbxy + b^2y^2 = 0$$

$$b^2y^2 + 2hbxy = -abx^2$$

To make L.H.S. complete square we add  $h^2x^2$  to both sides.

$$b^2y^2 + 2hbxy + h^2x^2 = h^2x^2 - abx^2$$

$$(by + hx)^2 = (h^2 - ab)x^2$$

$$(by + hx)^2 = \left(\sqrt{h^2 - ab}\right)^2 x^2, \text{ as } h^2 - ab \geq 0$$

$$(by + hx)^2 - \left(\sqrt{h^2 - ab}\right)^2 x^2 = 0$$

$$(by + hx + \sqrt{h^2 - ab}x)(by + hx - \sqrt{h^2 - ab}x) = 0$$

$$[(h + \sqrt{h^2 - ab})x + by] \times [(h - \sqrt{h^2 - ab})x + by] = 0$$

Which is the combined equation of lines  $(h + \sqrt{h^2 - ab})x + by = 0$  and  $(h - \sqrt{h^2 - ab})x + by = 0$ .

As  $b \neq 0$ , we can write these equations in the form  $y = m_1x$  and  $y = m_2x$ , where  $m_1 = \frac{-h - \sqrt{h^2 - ab}}{b}$  and  $m_2 = \frac{-h + \sqrt{h^2 - ab}}{b}$ .

We observe that these lines pass through the origin.

Therefore equation  $abx^2 + 2hbxy + by^2 = 0$  represents a pair of lines passing through the origin if  $h^2 - ab \geq 0$ .

### Remarks:

- 1) If  $h^2 - ab > 0$  then line represented by (1) are distinct.
- 2) If  $h^2 - ab = 0$  then lines represented by (1) are coincident.
- 3) If  $h^2 - ab < 0$  then equation (1) does not represent a pair of lines.
- 4) If  $b = 0$  then one of the lines is the Y - axis, whose slope is not defined and the slope of the other line is  $-\frac{a}{2h}$  (provided that  $h \neq 0$ ).

- 5) If  $h^2 - ab \geq 0$  and  $b \neq 0$  then slopes of the lines are  $m_1 = \frac{-h - \sqrt{h^2 - ab}}{b}$  and

$$m_2 = \frac{-h + \sqrt{h^2 - ab}}{b}$$

Their sum is  $m_1 + m_2 = -\frac{2h}{b}$  and product is  $m_1 m_2 = \frac{a}{b}$

The quadratic equation in  $m$  whose roots are  $m_1$  and  $m_2$  is given by

$$m^2 - (m_1 + m_2)m + m_1 m_2 = 0$$

$$\therefore m^2 - \left(-\frac{2h}{b}\right)m + \frac{a}{b} = 0$$

$$bm^2 + 2hm + a = 0 \quad \dots\dots (2)$$

Equation (2) is called the **auxiliary** equation of equation (1). Roots of equation (2) are slopes of lines represented by equation (1).



### Solved Examples

**Ex. 1)** Show that lines represented by equation  $x^2 - 2xy - 3y^2 = 0$  are distinct.

**Solution :** Comparing equation  $x^2 - 2xy - 3y^2 = 0$  with  $ax^2 + 2hxy + by^2 = 0$ , we get

$$a = 1, h = -1 \text{ and } b = -3.$$

$$\begin{aligned} h^2 - ab &= (-1)^2 - (1)(-3) \\ &= 1 + 3 \\ &= 4 > 0 \end{aligned}$$

As  $h^2 - ab > 0$ , lines represented by equation  $x^2 - 2xy - 3y^2 = 0$  are distinct.

**Ex. 2)** Show that lines represented by equation  $x^2 - 6xy + 9y^2 = 0$  are coincident.

**Solution :** Comparing equation  $x^2 - 6xy + 9y^2 = 0$  with  $ax^2 + 2hxy + by^2 = 0$ , we get

$$a = 1, h = -3 \text{ and } b = 9.$$

$$\begin{aligned} h^2 - ab &= (-3)^2 - (1)(9) \\ &= 9 - 9 = 0 \end{aligned}$$

As  $h^2 - ab = 0$ , lines represented by equation  $x^2 - 6xy + 9y^2 = 0$  are coincident.

**Ex. 3)** Find the sum and the product of slopes of lines represented by  $x^2 + 4xy - 7y^2 = 0$ .

**Solution :** Comparing equation  $x^2 + 4xy - 7y^2 = 0$  with  $ax^2 + 2hxy + by^2 = 0$ , we get  $a = 1, h = 2$  and  $b = -7$ .

If  $m_1$  and  $m_2$  are slopes of lines represented by this equation then

$$m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b}.$$

$$\therefore m_1 + m_2 = \frac{-4}{-7} = \frac{4}{7} \text{ and } m_1 m_2 = \frac{1}{-7} = -\frac{1}{7}$$

$$\text{Their sum is } \frac{4}{7} \text{ and products is } -\frac{1}{7}$$

**Ex. 4)** Find the separate equations of lines represented by

i)  $x^2 - 4y^2 = 0$

ii)  $3x^2 - 7xy + 4y^2 = 0$

iii)  $x^2 + 2xy - y^2 = 0$

iv)  $5x^2 - 3y^2 = 0$

**Solution :** i)  $x^2 - 4y^2 = 0$

$$\therefore (x - 2y)(x + 2y) = 0$$

Required separate equations are

$$x - 2y = 0 \text{ and } x + 2y = 0$$

ii)  $3x^2 - 7xy + 4y^2 = 0$

$$\therefore 3x^2 - 3xy - 4xy + 4y^2 = 0$$

$$\therefore 3x(x - y) - 4y(x - y) = 0$$

$$\therefore (x - y)(3x - 4y) = 0$$

Required separate equations are

$$x - y = 0 \text{ and } 3x - 4y = 0$$

$$\text{iii) } x^2 + 2xy - y^2 = 0$$

The corresponding auxiliary equation is  $bm^2 + 2hm + a = 0$

$$\therefore -m^2 + 2m + 1 = 0$$

$$m^2 - 2m - 1 = 0$$

$$\therefore m = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2}$$

$$= 1 \pm \sqrt{2}$$

Slopes of these lines are

$$m_1 = 1 + \sqrt{2} \text{ and } m_2 = 1 - \sqrt{2}$$

$\therefore$  Required separate equations are

$$y = m_1 x \text{ and } y = m_2 x$$

$$\therefore y = (1 + \sqrt{2})x \text{ and } y = (1 - \sqrt{2})x$$

$$\therefore (1 + \sqrt{2})x - y = 0 \text{ and } (1 - \sqrt{2})x - y = 0$$

$$\text{iv) } 5x^2 - 3y^2 = 0$$

$$\therefore (\sqrt{5}x)^2 - (\sqrt{3}y)^2 = 0$$

$$\therefore (\sqrt{5}x - \sqrt{3}y)(\sqrt{5}x + \sqrt{3}y) = 0$$

$\therefore$  Required separate equations are

$$\sqrt{5}x - \sqrt{3}y = 0 \text{ and } \sqrt{5}x + \sqrt{3}y = 0$$

**Ex. 5)** Find the value of  $k$  if  $2x + y = 0$  is one of the lines represented by  $3x^2 + kxy + 2y^2 = 0$ .

**Solution :** Slope of the line  $2x + y = 0$  is  $-2$

As  $2x + y = 0$  is one of the lines represented by  $3x^2 + kxy + 2y^2 = 0$ ,  $-2$  is a root of the auxiliary equation  $2m^2 + km + 3 = 0$

$$\therefore 2(-2)^2 + k(-2) + 3 = 0$$

$$\therefore 8 - 2k + 3 = 0$$

$$\therefore -2k + 11 = 0$$

$$\therefore 2k = 11 \qquad \therefore k = \frac{11}{2}.$$

**Alternative Method :** As  $2x + y = 0$  is one of the lines represented by  $3x^2 + kxy + 2y^2 = 0$ , co-ordinates of every point on the line  $2x + y = 0$  satisfy the equation  $3x^2 + kxy + 2y^2 = 0$ .

As  $(1, -2)$  is a point on the line  $2x + y = 0$ , it must satisfy the combined equation.

$$\therefore 3(1)^2 + k(1)(-2) + 2(-2)^2 = 0$$

$$\therefore -2k + 11 = 0$$

$$\therefore 2k = 11 \qquad \therefore k = \frac{11}{2}.$$

**Ex. 6)** Find the condition that the line  $3x - 2y = 0$  coincides with one of the lines represented by  $ax^2 + 2hxy + by^2 = 0$ .

**Solution :** The corresponding auxiliary equation is  $bm^2 + 2hm + a = 0$ .

As line  $3x - 2y = 0$  coincides with one of the lines given by  $ax^2 + 2hxy + by^2 = 0$ , its slope  $\frac{3}{2}$  is a root of the auxiliary equation.

$$\therefore \frac{3}{2} \text{ is a root of } bm^2 + 2hm + a = 0$$

$$\therefore b \left( \frac{3}{2} \right)^2 + 2h \left( \frac{3}{2} \right) + a = 0$$

$$\therefore \frac{9}{4}b + 3h + a = 0$$

$$\therefore 4a + 12h + 9b = 0 \text{ is the required condition.}$$

**Ex.7)** Find the combined equation of the pair of lines passing through the origin and perpendicular to the lines represented by  $3x^2 + 2xy - y^2 = 0$ .

**Solution :** Let  $m_1$  and  $m_2$  are slopes of lines represented by  $3x^2 + 2xy - y^2 = 0$ .

$$\therefore m_1 + m_2 = -\frac{2h}{b} = \frac{-2}{-1} = 2$$

$$\text{and } m_1 m_2 = \frac{a}{b} = \frac{3}{-1} = -3$$

Now required lines are perpendicular to given lines.

$$\therefore \text{ Their slopes are } -\frac{1}{m_1} \text{ and } -\frac{1}{m_2}$$

And required lines pass through the origin.

$$\therefore \text{ Their equations are } y = -\frac{1}{m_1}x \text{ and } y = -\frac{1}{m_2}x$$

$$\therefore m_1 y = -x \text{ and } m_2 y = -x$$

$$\therefore x + m_1 y = 0 \text{ and } x + m_2 y = 0$$

Their combined equation is  $(x + m_1 y)(x + m_2 y) = 0$

$$\therefore x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$$

$$\therefore x^2 + (2)xy + (-3)y^2 = 0$$

$$\therefore x^2 + 2xy - 3y^2 = 0$$

**Ex.8)** Find the value of  $k$ , if slope of one of the lines represented by  $4x^2 + kxy + y^2 = 0$  is four times the slope of the other line.

**Solution :** Let slopes of the lines represented by  $4x^2 + kxy + y^2 = 0$  be  $m$  and  $4m$ ,

their sum is  $m + 4m = 5m$

$$\text{But their sum is } \frac{-2h}{b} = \frac{-k}{1} = -k$$

$$\therefore 5m = -k$$

$$\therefore m = \frac{-k}{5} \dots (1)$$

Now their product is  $(m)(4m) = 4m^2$

But their product is  $\frac{a}{b} = \frac{4}{1} = 4$

$$\therefore 4m^2 = 4$$

$$\therefore m^2 = 1 \dots (2)$$

From (1) and (2), we get

$$\left(\frac{-k}{5}\right)^2 = 1$$

$$\therefore k^2 = 25$$

$$\therefore k = \pm 5$$



### Exercise 4.1

**1) Find the combined equation of the following pairs of lines:**

i)  $2x + y = 0$  and  $3x - y = 0$

ii)  $x + 2y - 1 = 0$  and  $x - 3y + 2 = 0$

iii) Passing through (2,3) and parallel to the co-ordinate axes.

iv) Passing through (2,3) and perpendicular to lines  $3x + 2y - 1 = 0$  and  $x - 3y + 2 = 0$

v) Passing through (-1,2), one is parallel to  $x + 3y - 1 = 0$  and the other is perpendicular to  $2x - 3y - 1 = 0$ .

**2) Find the separate equations of the lines represented by following equations:**

i)  $3y^2 + 7xy = 0$

ii)  $5x^2 - 9y^2 = 0$

iii)  $x^2 - 4xy = 0$

iv)  $3x^2 - 10xy - 8y^2 = 0$

v)  $3x^2 - 2\sqrt{3}xy - 3y^2 = 0$

vi)  $x^2 + 2(\operatorname{cosec} \alpha)xy + y^2 = 0$

vii)  $x^2 + 2xy \tan \alpha - y^2 = 0$

**3) Find the combined equation of a pair of lines passing through the origin and perpendicular to the lines represented by following equations :**

i)  $5x^2 - 8xy + 3y^2 = 0$

ii)  $5x^2 + 2xy - 3y^2 = 0$

iii)  $xy + y^2 = 0$

iv)  $3x^2 - 4xy = 0$

**4) Find  $k$  if,**

i) the sum of the slopes of the lines represented by  $x^2 + kxy - 3y^2 = 0$  is twice their product.

ii) slopes of lines represent by  $3x^2 + kxy - y^2 = 0$  differ by 4.

iii) slope of one of the lines given by  $kx^2 + 4xy - y^2 = 0$  exceeds the slope of the other by 8.



- 5) Find the condition that :
- i) the line  $4x + 5y = 0$  coincides with one of the lines given by  $ax^2 + 2hxy + by^2 = 0$ .
  - ii) the line  $3x + y = 0$  may be perpendicular to one of the lines given by  $ax^2 + 2hxy + by^2 = 0$ .
- 6) If one of the lines given by  $ax^2 + 2hxy + by^2 = 0$  is perpendicular to  $px + qy = 0$  then show that  $ap^2 + 2hpq + by^2 = 0$ .
- 7) Find the combined equation of the pair of lines passing through the origin and making an equilateral triangle with the line  $y = 3$ .
- 8) If slope of one of the lines given by  $ax^2 + 2hxy + by^2 = 0$  is four times the other then show that  $16h^2 = 25ab$ .
- 9) If one of the lines given by  $ax^2 + 2hxy + by^2 = 0$  bisects an angle between co-ordinate axes then show that  $(a+b)^2 = 4h^2$ .

### 4.3 Angle between lines represented by $ax^2 + 2hxy + by^2 = 0$ :

If we know slope of a line then we can find the angles made by the line with the co-ordinate axes. In equation  $ax^2 + 2hxy + by^2 = 0$  if  $b = 0$  then one of the lines is the Y - axis. Using the slope of the other line we can find the angle between them. In the following discussion we assume that  $b \neq 0$ , so that slopes of both lines will be defined.

If  $m_1$  and  $m_2$  are slopes of these lines then  $m_1 m_2 = -\frac{a}{b}$

We know that lines having slopes  $m_1$  and  $m_2$  are perpendicular to each other if and only if  $m_1 m_2 = -1$ .

$$\therefore \frac{a}{b} = -1$$

$$\therefore a = -b$$

$$\therefore a + b = 0$$

Thus lines represented by  $ax^2 + 2hxy + by^2 = 0$  are perpendicular to each other if and only if  $a + b = 0$ .

If lines are **not perpendicular** to each other then the acute angle between them can be obtained by using the following theorem.

**Theorem 4.4 :** The acute angle  $\theta$  between the lines represented by  $ax^2 + 2hxy + by^2 = 0$  is given by

$$\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

**Proof :** Let  $m_1$  and  $m_2$  be slopes of lines represented by the equation  $ax^2 + 2hxy + by^2 = 0$ .

$$\therefore m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b}$$

$$\therefore (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$$

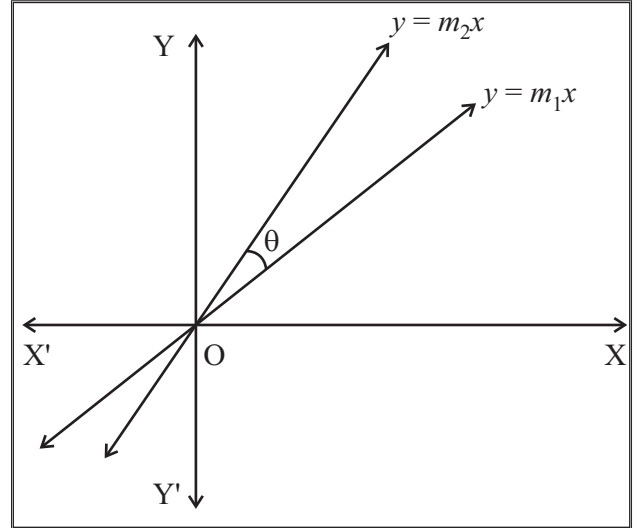
$$= \left( \frac{2h}{b} \right)^2 - 4 \left( \frac{a}{b} \right)$$

$$= \frac{4h^2}{b^2} - \frac{4ab}{b^2}$$

$$\begin{aligned}
 &= \frac{4h^2 - 4ab}{b^2} \\
 &= \frac{4(h^2 - ab)}{b^2} \\
 \therefore m_1 - m_2 &= \pm \frac{2\sqrt{h^2 - ab}}{b}
 \end{aligned}$$

As  $\theta$  is the acute angle between the lines,

$$\begin{aligned}
 \tan \theta &= \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \\
 &= \left| \frac{\pm \frac{2\sqrt{h^2 - ab}}{b}}{1 + \frac{a}{b}} \right| \\
 &= \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|
 \end{aligned}$$



**Figure 4.4**

**Remark :** Lines represented by  $ax^2 + 2hxy + by^2 = 0$  are coincident if and only if  $m_1 = m_2$

$$\begin{aligned}
 \therefore m_1 - m_2 &= 0 \\
 \therefore \frac{2\sqrt{h^2 - ab}}{b} &= 0 \\
 \therefore h^2 - ab &= 0 \\
 \therefore h^2 &= ab
 \end{aligned}$$

Lines represented by  $ax^2 + 2hxy + by^2 = 0$  are coincident if and only if  $h^2 = ab$ .



### Solved Examples

**Ex.1)** Show that lines represented by  $3x^2 - 4xy - 3y^2 = 0$  are perpendicular to each other.

**Solution :** Comparing given equation with  $ax^2 + 2hxy + by^2 = 0$  we get  $a = 3$ ,  $h = -2$  and  $b = -3$ .  
As  $a + b = 3 + (-3) = 0$ , lines represented by  $3x^2 - 4xy - 3y^2 = 0$  are perpendicular to each other.

**Ex. 2)** Show that lines represented by  $x^2 + 4xy + 4y^2 = 0$  are coincident.

**Solution :** Comparing given equation with  $ax^2 + 2hxy + by^2 = 0$ , we get  $a = 1$ ,  $h = 2$  and  $b = 4$ .

$$\begin{aligned}
 \text{As, } h^2 - ab &= (2)^2 - (1)(4) \\
 &= 4 - 4 = 0
 \end{aligned}$$

$\therefore$  Lines represented by  $x^2 + 4xy + 4y^2 = 0$  are coincident.

**Ex.3)** Find the acute angle between lines represented by:

i)  $x^2 + xy = 0$

ii)  $x^2 - 4xy + y^2 = 0$

iii)  $3x^2 + 2xy - y^2 = 0$

iv)  $2x^2 - 6xy + y^2 = 0$

v)  $xy + y^2 = 0$

**Solution :**

- i) Comparing equation  $x^2 + xy = 0$  with  $ax^2 + 2hxy + by^2 = 0$ , we get  $a = 1$ ,  $h = \frac{1}{2}$  and  $b = 0$ .  
Let  $\theta$  be the acute angle between them.

$$\therefore \tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| = \left| \frac{2\sqrt{\frac{1}{4} - 0}}{1} \right| = 1$$

$$\therefore \theta = 45^\circ = \frac{\pi}{4}$$

- ii) Comparing equation  $x^2 - 4xy + y^2 = 0$  with  $ax^2 + 2hxy + by^2 = 0$ , we get  $a = 1$ ,  $h = -2$  and  $b = 1$ .  
Let  $\theta$  be the acute angle between them.

$$\therefore \tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

$$= \left| \frac{2\sqrt{4 - 1}}{2} \right| = \sqrt{3}$$

$$\therefore \theta = 60^\circ = \frac{\pi}{3}$$

- iii) Comparing equation  $3x^2 + 2xy - y^2 = 0$  with  $ax^2 + 2hxy + by^2 = 0$ , we get  $a = 3$ ,  $h = 1$  and  $b = -1$ .  
Let  $\theta$  be the acute angle between them.

$$\therefore \tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

$$= \left| \frac{2\sqrt{1 + 3}}{2} \right| = 2.$$

$$\therefore \theta = \tan^{-1}(2)$$

- iv) Comparing equation  $2x^2 - 6xy + y^2 = 0$  with  $ax^2 + 2hxy + by^2 = 0$ , we get  $a = 2$ ,  $h = -3$  and  $b = 1$ .  
Let  $\theta$  be the acute angle between them.

$$\therefore \tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

$$= \left| \frac{2\sqrt{9 - 2}}{3} \right| = \frac{2\sqrt{7}}{3} \therefore \theta = \tan^{-1} \left( \frac{2\sqrt{7}}{3} \right).$$

v) Comparing equation  $xy + y^2 = 0$  with  $ax^2 + 2hxy + by^2 = 0$ , we get  $a = 0$ ,  $h = \frac{1}{2}$  and  $b = 1$ .

Let  $\theta$  be the acute angle between them.

$$\begin{aligned}\therefore \tan \theta &= \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| \\ &= \left| \frac{2\sqrt{\frac{1}{4} - 0}}{1} \right| = 1\end{aligned}$$

$$\therefore \theta = 45^\circ = \frac{\pi}{4}.$$

**Ex.4)** Find the combined equation of lines passing through the origin and making angle  $\frac{\pi}{6}$  with the line  $3x + y - 6 = 0$ .

**Solution :** Let  $m$  be the slope of one of the lines which make angle  $\frac{\pi}{6}$  with the line  $3x + y - 6 = 0$ . Slope of the given line is  $-3$ .

$$\therefore \tan \frac{\pi}{6} = \left| \frac{m - (-3)}{1 + m(-3)} \right|$$

$$\therefore \frac{1}{\sqrt{3}} = \left| \frac{m + 3}{1 - 3m} \right|$$

$$\therefore (1 - 3m)^2 = 3(m + 3)^2$$

$$\therefore 9m^2 - 6m + 1 = 3(m^2 + 6m + 9)$$

$$\therefore 9m^2 - 6m + 1 = 3m^2 + 18m + 27$$

$$\therefore 6m^2 - 24m - 26 = 0$$

$$\therefore 3m^2 - 12m - 13 = 0$$

This is the auxiliary equation of the required combined equation.

The required combined equation is  $-13x^2 - 12xy + 3y^2 = 0$

$$\therefore 13x^2 + 12xy - 3y^2 = 0$$

**Ex. 5)** Find the combined equation of lines passing through the origin and each of which making angle  $60^\circ$  with the X - axis.

**Solution :**

Let  $m$  be the slope of one of the required lines.

The slope of the X - axis is 0. As required lines make angle  $60^\circ$  with the X - axis,

$$\tan 60^\circ = \left| \frac{m - 0}{1 + (m)(0)} \right|$$

$$\therefore \sqrt{3} = |m|$$

$$\therefore m^2 = 3$$

$$\therefore m^2 + 0m - 3 = 0 \text{ is the auxiliary equation.}$$

$\therefore$  The required combined equation is

$$-3x^2 + 0xy + y^2 = 0$$

$$\therefore 3x^2 - y^2 = 0$$

**Alternative Method:** As required lines make angle  $60^\circ$  with the X - axis, their inclination are  $60^\circ$  and  $120^\circ$ . Hence their slopes are  $\sqrt{3}$  and  $-\sqrt{3}$ .

Lines pass through the origin. Their equations are  $y = \sqrt{3}x$  and  $y = -\sqrt{3}x$

$$\therefore \sqrt{3}x - y = 0 \text{ and } \sqrt{3}x + y = 0$$

Their combined equation is  $(\sqrt{3}x - y)(\sqrt{3}x + y) = 0$

$$\therefore 3x^2 - y^2 = 0$$



### Exercise 4.2

- 1) Show that lines represented by  $3x^2 - 4xy - 3y^2 = 0$  are perpendicular to each other.
- 2) Show that lines represented by  $x^2 + 6xy + gy^2 = 0$  are coincident.
- 3) Find the value of  $k$  if lines represented by  $kx^2 + 4xy - 4y^2 = 0$  are perpendicular to each other.
- 4) Find the measure of the acute angle between the lines represented by:
  - i)  $3x^2 - 4\sqrt{3}xy + 3y^2 = 0$
  - ii)  $4x^2 + 5xy + 4y^2 = 0$
  - iii)  $2x^2 + 7xy + 3y^2 = 0$
  - iv)  $(a^2 - 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$
- 5) Find the combined equation of lines passing through the origin each of which making an angle of  $30^\circ$  with the line  $3x + 2y - 11 = 0$
- 6) If the angle between lines represented by  $ax^2 + 2hxy + by^2 = 0$  is equal to the angle between lines represented by  $2x^2 - 5xy + 3y^2 = 0$  then show that  $100(h^2 - ab) = (a + b)^2$ .
- 7) Find the combined equation of lines passing through the origin and each of which making angle  $60^\circ$  with the Y- axis.

### 4.4 General Second Degree Equation in x and y:

Equation of the form  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ , where at least one of  $a, b, h$  is not zero, is called a general second degree equation in  $x$  and  $y$ .

**Theorem 4.5 :** The combined equation of two lines is a general second degree equation in  $x$  and  $y$ .

**Proof:** Let  $u \equiv a_1x + b_1y + c_1$  and  $v \equiv a_2x + b_2y + c_2$ . Equations  $u = 0$  and  $v = 0$  represent lines. Their combined equation is  $uv = 0$ .

$$\therefore (a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$$

$$a_1a_2x^2 + a_1b_2xy + a_1c_2x + b_1a_2xy + b_1b_2y^2 + b_1c_2y + c_1a_2x + c_1b_2y + c_1c_2 = 0$$

$$\text{Writing } a_1a_2 = a, b_1b_2 = b, a_1b_2 + a_2b_1 = 2h, a_1c_2 + a_2c_1 = 2g, b_1c_2 + b_2c_1 = 2f, c_1c_2 = c,$$

we get,  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ , which is the general equation of degree two in  $x$  and  $y$ .

**Remark :** The converse of the above theorem is **not true**. Every general second degree equation in  $x$  and  $y$  **need not** represent a pair of lines. For example  $x^2 + y^2 = 25$  is a general second degree equation in  $x$  and  $y$  but it does not represent a pair of lines. It represents a circle.

Equation  $x^2 + y^2 - 4x + 6y + 13 = 0$  is also a general second degree equation which does not represent a pair of lines. How to identify that whether the given equation represents a pair of lines or not?

**4.4.1** The necessary conditions for a general second degree equation.

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  to represent a pair of lines are:

i)  $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$                       ii)  $h^2 - ab \geq 0$

**Remarks :**

If equation  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  represents a pair of lines then

1) These lines are parallel to the line represented by  $ax^2 + 2hxy + by^2 = 0$

2) The acute angle between them is given by  $\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a+b} \right|$

3) Condition for lines to be perpendicular to each other is  $a + b = 0$ .

4) Condition for lines to be parallel to each other is  $h^2 - ab = 0$ .

5) Condition for lines to intersect each other is  $h^2 - ab > 0$  and the co-ordinates of their point

of intersection are  $\left( \frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right)$

6) The expression  $abc + 2fgh - af^2 - bg^2 - ch^2$  is the expansion of the determinant  $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

7) The joint equation of the bisector of the angle between the lines represented by

$ax^2 + 2hxy + by^2 = 0$  is  $hx^2 - (a-b)xy - hy^2 = 0$ . Here coefficient of  $x^2$  + coefficient of  $y^2 = 0$ .

Hence bisectors are perpendicular to each other



**Solved Examples**

**Ex.1)** Show that equation  $x^2 - 6xy + 5y^2 + 10x - 14y + 9 = 0$  represents a pair of lines. Find the acute angle between them. Also find the point of their intersection.

**Solution:** We have  $x^2 - 6xy + 5y^2 = (x - 5y)(x - y)$

Suppose  $x^2 - 6xy + 5y^2 + 10x - 14y + 9 = (x - 5y + c)(x - y + k)$

$$\therefore x^2 - 6xy + 5y^2 + 10x - 14y + 9 = x^2 - 6xy + 5y^2 + (c + k)x - (c + 5k)y + ck$$

$$\therefore c + k = 10, c + 5k = 14 \text{ and } ck = 9$$

We observe that  $c = 9$  and  $k = 1$  satisfy all three equations.

$\therefore$  Given general equation can be factorized as  $(x - 5y + 9)(x - y + 1) = 0$

$\therefore$  Given equation represents a pair of intersecting lines.

The acute angle between them is given by

$$\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a+b} \right| = \left| \frac{2\sqrt{(-3)^2 - (1)(5)}}{1+5} \right| = \frac{2}{3}$$

$$\therefore \theta = \tan^{-1} \left( \frac{2}{3} \right)$$

Their point of intersection is given by

$$\left( \frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right) = \left( \frac{21 - 25}{5 - 9}, \frac{-15 + 7}{5 - 9} \right) = (1, 2)$$

**Remark :**

Note that condition  $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$  is not sufficient for equation to represent a pair of lines. We can't use this condition to show that given equation represents a pair of lines.

**Ex.2)** Find the value of  $k$  if the equation  $2x^2 + 4xy - 2y^2 + 4x + 8y + k = 0$  represents a pair of lines.

**Solution:** Comparing given equation with  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  we get

$$a = 2, b = -2, c = k, f = 4, g = 2, h = 2.$$

As given equation represents a pair of lines, it must satisfy the necessary condition.

$$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\therefore (2)(-2)(k) + 2(4)(2)(2) - 2(4)^2 - (-2)(2)^2 - (k)(2)^2 = 0$$

$$\therefore -4k + 32 - 32 + 8 - 4k = 0$$

$$\therefore 8k = 8$$

$$\therefore k = 1.$$

**Ex.3)** Find  $p$  and  $q$  if the equation  $2x^2 + 4xy - py^2 + 4x + qy + 1 = 0$  represents a pair of perpendicular lines.

**Solution:** Comparing given equation with  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  we get

$$a = 2, b = -p, c = 1, f = \frac{q}{2}, g = 2, h = 2$$

As lines are perpendicular to each other,  $a + b = 0$

$$\therefore 2 + (-p) = 0$$

$$\therefore p = 2$$

As given equation represents a pair of lines, it must satisfy the necessary condition.

$$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\therefore (2)(-p)(1) + 2\left(\frac{q}{2}\right)(2)(2) - 2\left(\frac{q}{2}\right)^2 - (-p)(2)^2 - 1(2)^2 = 0$$

$$\therefore -2p + 4q - \frac{q^2}{2} + 4p - 4 = 0$$

$$\therefore 2p + 4q - \frac{q^2}{2} - 4 = 0 \quad \dots(1)$$

substituting  $p = 2$  in (1), we get

$$\therefore 2(2) + 4q - \frac{q^2}{2} - 4 = 0$$

$$\therefore 4q - \frac{q^2}{2} = 0 \therefore 8q - q^2 = 0$$

$$\therefore q(8 - q) = 0$$

$$\therefore q = 0 \text{ or } q = 8$$

**Ex.4)**  $\Delta OAB$  is formed by lines  $x^2 - 4xy + y^2 = 0$  and the line  $x + y - 2 = 0$ . Find the equation of the median of the triangle drawn from O.

**Solution :** Let the co-ordinates of A and B be  $(x_1, y_1)$  and  $(x_2, y_2)$  respectively.

The midpoint of segment AB is

$$P \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

The co-ordinates of A and B can be obtained

by solving equations  $x + y - 2 = 0$  and

$x^2 - 4xy + y^2 = 0$  simultaneously.

put  $y = 2 - x$  in  $x^2 - 4xy + y^2 = 0$ .

$$x^2 - 4x(2 - x) + (2 - x)^2 = 0$$

$$\therefore 6x^2 - 12x + 4 = 0$$

$$\therefore 3x^2 - 6x + 2 = 0$$

$x_1$  and  $x_2$  are roots of this equation.

$$x_1 + x_2 = -\frac{-6}{3} = 2$$

$$\therefore \frac{x_1 + x_2}{2} = 1$$

The x co-ordinate of P is 1.

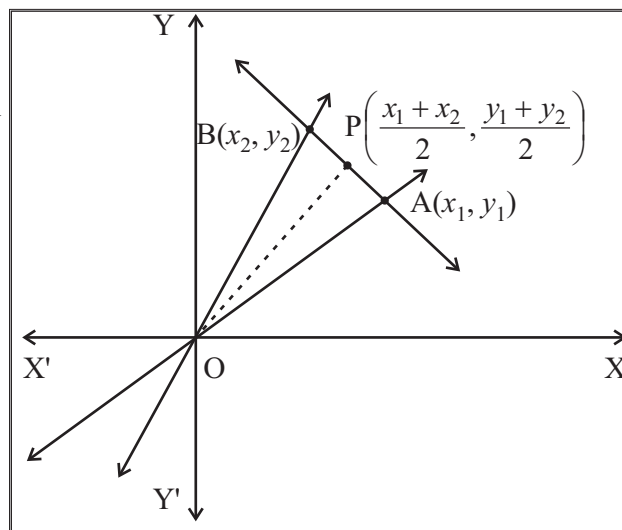
As P lies on the line  $x + y - 2 = 0$

$$\therefore 1 + y - 2 = 0 \quad \therefore y = 1$$

$\therefore$  Co-ordinates of P are (1,1).

The equation of the median OP is  $\frac{y-0}{1-0} = \frac{x-0}{1-0}$

$$\therefore y = x \quad \therefore x - y = 0.$$



**Figure 4.5**



### Exercise 4.3

- Find the joint equation of the pair of lines:
  - Through the point  $(2, -1)$  and parallel to lines represented by  $2x^2 + 3xy - 9y^2 = 0$
  - Through the point  $(2, -3)$  and parallel to lines represented by  $x^2 + xy - y^2 = 0$
- Show that equation  $x^2 + 2xy + 2y^2 + 2x + 2y + 1 = 0$  does not represent a pair of lines.
- Show that equation  $2x^2 - xy - 3y^2 - 6x + 19y - 20 = 0$  represents a pair of lines.
- Show the equation  $2x^2 + xy - y^2 + x + 4y - 3 = 0$  represents a pair of lines. Also find the acute angle between them.
- Find the separate equation of the lines represented by the following equations :
  - $(x - 2)^2 - 3(x - 2)(y + 1) + 2(y + 1)^2 = 0$
  - $10(x + 1)^2 + (x + 1)(y - 2) - 3(y - 2)^2 = 0$
- Find the value of  $k$  if the following equations represent a pair of lines :
  - $3x^2 + 10xy + 3y^2 + 16y + k = 0$
  - $kxy + 10x + 6y + 4 = 0$
  - $x^2 + 3xy + 2y^2 + x - y + k = 0$



- 7) Find  $p$  and  $q$  if the equation  $px^2 - 8xy + 3y^2 + 14x + 2y + q = 0$  represents a pair of perpendicular lines.
- 8) Find  $p$  and  $q$  if the equation  $2x^2 + 8xy + py^2 + qx + 2y - 15 = 0$  represents a pair of parallel lines.
- 9) Equations of pairs of opposite sides of a parallelogram are  $x^2 - 7x + 6 = 0$  and  $y^2 - 14y + 40 = 0$ . Find the joint equation of its diagonals.
- 10)  $\Delta OAB$  is formed by lines  $x^2 - 4xy + y^2 = 0$  and the line  $2x + 3y - 1 = 0$ . Find the equation of the median of the triangle drawn from  $O$ .
- 11) Find the co-ordinates of the points of intersection of the lines represented by  $x^2 - y^2 - 2x + 1 = 0$ .



### Let's remember!

- An equation which represents two lines is called the combined equation of those two lines.
- The equation  $uv = 0$  represents the combined equation of lines  $u = 0$  and  $v = 0$ .
- The sum of the indices of all variables in a term is called the degree of the term.
- An equation in which the degree of every term is same, is called a homogeneous equation.
- The combined equation of a pair of lines passing through the origin is a homogeneous equation of degree two in  $x$  and  $y$ .
- Every homogeneous equation of degree two need not represent a pair of lines.
- A homogeneous equation of degree two in  $x$  and  $y$ ,  $ax^2 + 2hxy + by^2 = 0$  represents a pair of lines passing through the origin if  $h^2 - ab \geq 0$ .
- If  $h^2 - ab > 0$  then lines are distinct.
- If  $h^2 - ab = 0$  then lines are coincident.
- Slopes of these lines are  $m_1 = \frac{-h - \sqrt{h^2 - ab}}{b}$  and  $m_2 = \frac{-h + \sqrt{h^2 - ab}}{b}$
- Their sum is,  $m_1 + m_2 = -\frac{2h}{b}$  and product is,  $m_1 m_2 = \frac{a}{b}$
- The quadratic equation in  $m$  whose roots are  $m_1$  and  $m_2$  is given by  $bm^2 + 2hm + a = 0$ , called the **auxiliary** equation.
- Lines represented by  $ax^2 + 2hxy + by^2 = 0$  are perpendicular to each other if and only if  $a + b = 0$ .
- Lines represented by  $ax^2 + 2hxy + by^2 = 0$  are coincident if and only if  $h^2 - ab = 0$ .
- The acute angle  $\theta$  between the lines represented by  $ax^2 + 2hxy + by^2 = 0$  is given by
 
$$\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$
- Equation of the form  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  is called a general second degree equation in  $x$  and  $y$ .
- The combined equation of two lines is a general second degree equation in  $x$  and  $y$ .
- The necessary conditions for a general second degree equation  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  to represent a pair of lines are:
  - i)  $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
  - ii)  $h^2 - ab \geq 0$

- The expression  $abc + 2fgh - af^2 - bg^2 - ch^2$  is the expansion of the determinant  $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$
- If equation  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  represents a pair of lines then
  - These lines are parallel to the lines represented by  $ax^2 + 2hxy + by^2 = 0$ .
  - The acute angle between them is given by  $\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a+b} \right|$
  - Condition for lines to be perpendicular to each other is  $a + b = 0$ .
  - Condition for lines to be parallel to each other is  $h^2 - ab = 0$ .
  - Condition for lines to intersect each other is  $h^2 - ab \geq 0$  and the co-ordinates of their point of intersection are  $\left( \frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right)$

#### MISCELLANEOUS EXERCISE 4

##### I : Choose correct alternatives.

- If the equation  $4x^2 + hxy + y^2 = 0$  represents two coincident lines, then  $h =$  \_\_\_\_\_.
  - $\pm 2$
  - $\pm 3$
  - $\pm 4$
  - $\pm 5$
- If the lines represented by  $kx^2 - 3xy + 6y^2 = 0$  are perpendicular to each other then \_\_\_\_\_.
  - $k = 6$
  - $k = -6$
  - $k = 3$
  - $k = -3$
- Auxiliary equation of  $2x^2 + 3xy - 9y^2 = 0$  is \_\_\_\_\_.
  - $2m^2 + 3m - 9 = 0$
  - $9m^2 - 3m - 2 = 0$
  - $2m^2 - 3m + 9 = 0$
  - $-9m^2 - 3m + 2 = 0$
- The difference between the slopes of the lines represented by  $3x^2 - 4xy + y^2 = 0$  is \_\_\_\_\_.
  - 2
  - 1
  - 3
  - 4
- If the two lines  $ax^2 + 2hxy + by^2 = 0$  make angles  $\alpha$  and  $\beta$  with X-axis, then  $\tan(\alpha + \beta) =$  \_\_\_\_\_.
  - $\frac{h}{a+b}$
  - $\frac{h}{a-b}$
  - $\frac{2h}{a+b}$
  - $\frac{2h}{a-b}$
- If the slope of one of the two lines  $\frac{x^2}{a} + \frac{2xy}{h} + \frac{y^2}{b} = 0$  is twice that of the other, then  $ab:h^2 =$  \_\_\_\_\_.
  - 1 : 2
  - 2 : 1

- C) 8 : 9  
D) 9 : 8
- 7) The joint equation of the lines through the origin and perpendicular to the pair of lines  $3x^2 + 4xy - 5y^2 = 0$  is \_\_\_\_\_.
- A)  $5x^2 + 4xy - 3y^2 = 0$   
B)  $3x^2 + 4xy - 5y^2 = 0$   
C)  $3x^2 - 4xy + 5y^2 = 0$   
D)  $5x^2 + 4xy + 3y^2 = 0$
- 8) If acute angle between lines  $ax^2 + 2hxy + by^2 = 0$  is,  $\frac{\pi}{4}$  then  $4h^2 =$  \_\_\_\_\_.
- A)  $a^2 + 4ab + b^2$   
B)  $a^2 + 6ab + b^2$   
C)  $(a + 2b)(a + 3b)$   
D)  $(a - 2b)(2a + b)$
- 9) If the equation  $3x^2 - 8xy + qy^2 + 2x + 14y + p = 1$  represents a pair of perpendicular lines then the values of  $p$  and  $q$  are respectively \_\_\_\_\_.
- A) -3 and -7  
B) -7 and -3  
C) 3 and 7  
D) -7 and 3
- 10) The area of triangle formed by the lines  $x^2 + 4xy + y^2 = 0$  and  $x - y - 4 = 0$  is \_\_\_\_\_.
- A)  $\frac{4}{\sqrt{3}}$  Sq. units  
B)  $\frac{8}{\sqrt{3}}$  Sq. units  
C)  $\frac{16}{\sqrt{3}}$  Sq. units  
D)  $\frac{15}{\sqrt{3}}$  Sq. units
- 11) The combined equation of the co-ordinate axes is \_\_\_\_\_.
- A)  $x + y = 0$   
B)  $xy = k$   
C)  $xy = 0$   
D)  $x - y = k$
- 12) If  $h^2 = ab$ , then slope of lines  $ax^2 + 2hxy + by^2 = 0$  are in the ratio \_\_\_\_\_.
- A) 1 : 2  
B) 2 : 1  
C) 2 : 3  
D) 1 : 1
- 13) If slope of one of the lines  $ax^2 + 2hxy + by^2 = 0$  is 5 times the slope of the other, then  $5h^2 =$  \_\_\_\_\_.
- A)  $ab$   
B)  $2ab$   
C)  $7ab$   
D)  $9ab$
- 14) If distance between lines  $(x - 2y)^2 + k(x - 2y) = 0$  is 3 units, then  $k =$  \_\_\_\_\_.
- A)  $\pm 3$   
B)  $\pm 5\sqrt{5}$   
C) 0  
D)  $\pm 3\sqrt{5}$

## II. Solve the following.

- 1) Find the joint equation of lines:
  - i)  $x - y = 0$  and  $x + y = 0$
  - ii)  $x + y - 3 = 0$  and  $2x + y - 1 = 0$
  - iii) Passing through the origin and having slopes 2 and 3.
  - iv) Passing through the origin and having inclinations  $60^\circ$  and  $120^\circ$ .
  - v) Passing through (1,2) and parallel to the co-ordinate axes.
  - vi) Passing through (3,2) and parallel to the line  $x = 2$  and  $y = 3$ .

- vii) Passing through  $(-1, 2)$  and perpendicular to the lines  $x + 2y + 3 = 0$  and  $3x - 4y - 5 = 0$ .
- viii) Passing through the origin and having slopes  $1 + \sqrt{3}$  and  $1 - \sqrt{3}$
- ix) Which are at a distance of 9 units from the Y - axis.
- x) Passing through the point  $(3, 2)$ , one of which is parallel to the line  $x - 2y = 2$  and other is perpendicular to the line  $y = 3$ .
- xi) Passing through the origin and perpendicular to the lines  $x + 2y = 19$  and  $3x + y = 18$ .
- 2) Show that each of the following equation represents a pair of lines.
- $x^2 + 2xy - y^2 = 0$
  - $4x^2 + 4xy + y^2 = 0$
  - $x^2 - y^2 = 0$
  - $x^2 + 7xy - 2y^2 = 0$
  - $x^2 - 2\sqrt{3}xy - y^2 = 0$
- 3) Find the separate equations of lines represented by the following equations:
- $6x^2 - 5xy - 6y^2 = 0$
  - $x^2 - 4y^2 = 0$
  - $3x^2 - y^2 = 0$
  - $2x^2 + 2xy - y^2 = 0$
- 4) Find the joint equation of the pair of lines through the origin and perpendicular to the lines given by :
- $x^2 + 4xy - 5y^2 = 0$
  - $2x^2 - 3xy - 9y^2 = 0$
  - $x^2 + xy - y^2 = 0$
- 5) Find  $k$  if
- The sum of the slopes of the lines given by  $3x^2 + kxy - y^2 = 0$  is zero.
  - The sum of slopes of the lines given by  $x^2 + kxy - 3y^2 = 0$  is equal to their product.
  - The slope of one of the lines given by  $3x^2 - 4xy + 5y^2 = 0$  is 1.
  - One of the lines given by  $3x^2 - kxy + 5y^2 = 0$  is perpendicular to the  $5x + 3y = 0$ .
  - The slope of one of the lines given by  $3x^2 + 4xy + ky^2 = 0$  is three times the other.
  - The slopes of lines given by  $kx^2 + 5xy + y^2 = 0$  differ by 1.
  - One of the lines given by  $6x^2 + kxy + y^2 = 0$  is  $2x + y = 0$ .
- 6) Find the joint equation of the pair of lines which bisect angle between the lines given by  $x^2 + 3xy + 2y^2 = 0$
- 7) Find the joint equation of the pair of lines through the origin and making equilateral triangle with the line  $x = 3$ .
- 8) Show that the lines  $x^2 - 4xy + y^2 = 0$  and  $x + y = 10$  contain the sides of an equilateral triangle. Find the area of the triangle.
- 9) If the slope of one of the lines represented by  $ax^2 + 2hxy + by^2 = 0$  is three times the other then prove that  $3h^2 = 4ab$ .

- 10) Find the combined equation of the bisectors of the angles between the lines represented by  $5x^2 + 6xy - y^2 = 0$ .
- 11) Find  $a$  if the sum of slope of lines represented by  $ax^2 + 8xy + 5y^2 = 0$  is twice their product.
- 12) If line  $4x - 5y = 0$  coincides with one of the lines given by  $ax^2 + 2hxy + by^2 = 0$  then show that  $25a + 40h + 16b = 0$ .
- 13) Show that the following equations represent a pair of lines, find the acute angle between them.
- $9x^2 - 6xy + y^2 + 18x - 6y + 8 = 0$
  - $2x^2 + xy - y^2 + x - 4y + 3 = 0$
  - $(x - 3)^2 + (x - 3)(y - 4) - 2(y - 4)^2 = 0$
- 14) Find the combined equation of pair of lines through the origin each of which makes angle of  $60^\circ$  with the Y-axis.
- 15) If lines represented by  $ax^2 + 2hxy + by^2 = 0$  make angles of equal measures with the co-ordinate axes then show that  $a = \pm b$ .
- 16) Show that the combined equation of a pair of lines through the origin and each making an angle of  $\alpha$  with the line  $x + y = 0$  is  $x^2 + 2(\sec 2\alpha)xy + y^2 = 0$ .
- 17) Show that the line  $3x + 4y + 5 = 0$  and the lines  $(3x + 4y)^2 - 3(4x - 3y)^2 = 0$  form an equilateral triangle.
- 18) Show that lines  $x^2 - 4xy + y^2 = 0$  and  $x + y = \sqrt{6}$  form an equilateral triangle. Find its area and perimeter.
- 19) If the slope of one of the lines given by  $ax^2 + 2hxy + by^2 = 0$  is square of the other then show that  $a^2b + ab^2 + 8h^3 = 6abh$ .
- 20) Prove that the product of lengths of perpendiculars drawn from P  $(x_1, y_1)$  to the lines represented by  $ax^2 + 2hxy + by^2 = 0$  is  $\left| \frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}} \right|$
- 21) Show that the difference between the slopes of lines given by  $(\tan^2\theta + \cos^2\theta)x^2 - 2xy \tan \theta + (\sin^2 \theta)y^2 = 0$  is two.
- 22) Find the condition that the equation  $ay^2 + bxy + ex + dy = 0$  may represent a pair of lines.
- 23) If the lines given by  $ax^2 + 2hxy + by^2 = 0$  form an equilateral triangle with the line  $lx + my = 1$  then show that  $(3a + b)(a + 3b) = 4h^2$ .
- 24) If line  $x + 2 = 0$  coincides with one of the lines represented by the equation  $x^2 + 2xy + 4y + k = 0$  then show that  $k = -4$ .
- 25) Prove that the combined equation of the pair of lines passing through the origin and perpendicular to the lines represented by  $ax^2 + 2hxy + by^2 = 0$  is  $bx^2 - 2hxy + ay^2 = 0$
- 26) If equation  $ax^2 - y^2 + 2y + c = 1$  represents a pair of perpendicular lines then find  $a$  and  $c$ .

