6 Factorisation of Algebraic expressions

Let's recall.

In the previous standard we have learnt to factorise the expressions of the form $ax + ay$ and $a^2 - b^2$

For example, (1) $4xy + 8xy^2 = 4xy(1 + 2y)$

(2) $p^2 - 9q^2 = (p)^2 - (3q)^2 = (p + 3q)(p - 3q)$

Let's learn.

Factors of a quadratic trinomial

An expression of the form $ax^2 + bx + c$ is called a quadratic trinomial.

We know that $(x + a)(x + b) = x^2 + (a + b)x + ab$

$\therefore$ the factors of $x^2 + (a + b)x + ab$ are $(x + a)$ and $(x + b)$.

To find the factors of $x^2 + 5x + 6$, by comparing it with $x^2 + (a + b)x + ab$ we get, $a + b = 5$ and $ab = 6$. So, let us find the factors of 6 whose sum is 5. Then writing the trinomial in the form $x^2 + (a + b)x + ab$, find its factors.

$x^2 + 5x + 6 = x^2 + (3 + 2)x + 3 \times 2$ .... $x^2 + (a + b)x + ab$

$= \underline{x^2 + 3x} + \underline{2x + 2 \times 3}$ .... multiply $(3 + 2)$ by x, make two groups of the four terms obtained.

$= x(x + 3) + 2(x + 3) = (x + 3)(x + 2)$

Study the following examples to know how a given trinomial is factorised.

Ex. (1) Factorise : $2x^2 - 9x + 9$.

Solution: First we find the product of the coefficient of the square term and the constant term. Here the product is $2 \times 9 = 18$.

Now, find factors of 18 whose sum is -9, that is equal to the coefficient of the middle term.

$18 = (-6) \times (-3)$; $(-6) + (-3) = -9$

Write the term $-9x$ as $-6x - 3x$

$2x^2 - 9x + 9$

$= \underline{2x^2 - 6x} - \underline{3x + 9}$

$= 2x\,\underline{(x - 3)} - 3\underline{(x - 3)}$

$= (x - 3)(2x - 3)$

$\therefore\ 2x^2 - 9x + 9 = (x - 3)(2x - 3)$

Ex. (2) Factorise : $2x^2 + 5x - 18$.

Solution : $2x^2 + 5x - 18$

−36: +9, −4

$= \underline{2x^2 + 9x} \underline{- 4x - 18}$

$= x(2x + 9) - 2(2x + 9)$

$= (2x + 9)(x - 2)$

Ex. (3) Factorise : $x^2 - 10x + 21$.

solution: $x^2 - 10x + 21$

+21: −7, −3

$= \underline{x^2 - 7x} \underline{- 3x + 21}$

$= x(x - 7) - 3(x - 7)$

$= (x - 7)(x - 3)$

Ex. (4) Find the factors of $2y^2 - 4y - 30$.

Solution : $2y^2 - 4y - 30$

$= 2(y^2 - 2y - 15)$ .... taking out the common factor 2

$= 2(\underline{y^2 - 5y} + \underline{3y - 15})$ ....

−15: −5, +3

$= 2[y(y - 5) + 3(y - 5)]$

$= 2(y - 5)(y + 3)$

Practice Set 6.1

1. Factorise.

(1) $x^2 + 9x + 18$	(2) $x^2 - 10x + 9$	(3) $y^2 + 24y + 144$
(4) $5y^2 + 5y - 10$	(5) $p^2 - 2p - 35$	(6) $p^2 - 7p - 44$
(7) $m^2 - 23m + 120$	(8) $m^2 - 25m + 100$	(9) $3x^2 + 14x + 15$
(10) $2x^2 + x - 45$	(11) $20x^2 - 26x + 8$	(12) $44x^2 - x - 3$

Let's learn.

Factors of $a^3 + b^3$

We know that, $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, which we can write as

$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

Now, $a^3 + b^3 + 3ab(a + b) = (a + b)^3$.......... interchanging the sides.

$\therefore\ a^3 + b^3 = (a + b)^3 - 3ab(a + b) = [(a + b)(a + b)^2] - 3ab(a + b)$

$= (a + b)[(a + b)^2 - 3ab] = (a + b)\ (a^2 + 2ab + b^2 - 3ab)$

$= (a + b)\ (a^2 - ab + b^2)$

$\therefore\ \boxed{\mathbf{a^3 + b^3 = (a + b)\ (a^2 - ab + b^2)}}$

Lets us solve some examples using the above formula for factorising the addition of two cubes.

Ex. (1) $x^3 + 27y^3 = x^3 + (3y)^3$

$$= (x + 3y)\,[x^2 - x(3y) + (3y)^2]$$

$$= (x + 3y)\,[x^2 - 3xy + 9y^2]$$

Ex. (2) $8p^3 + 125q^3 = (2p)^3 + (5q)^3 = (2p + 5q)\,[(2p)^2 - 2p \times 5q + (5q)^2]$

$$= (2p + 5q)\,(4p^2 - 10pq + 25q^2)$$

Ex. (3) $m^3 + \frac{1}{64m^3} = m^3 + \left(\frac{1}{4m}\right)^3 = \left(m + \frac{1}{4m}\right)\left[m^2 - m \times \frac{1}{4m} + \left(\frac{1}{4m}\right)^2\right]$

$$= \left(m + \frac{1}{4m}\right)\left(m^2 - \frac{1}{4} + \frac{1}{16m^2}\right)$$

Ex. (4) $250p^3 + 432q^3 = 2(125p^3 + 216q^3)$

$$= 2\,[(5p)^3 + (6q)^3] = 2(5p + 6q)(25p^2 - 30pq + 36q^2)$$

Practice Set 6.2

1. Factorise. (1) $x^3 + 64y^3$ (2) $125p^3 + q^3$ (3) $125k^3 + 27m^3$ (4) $2l^3 + 432m^3$

(5) $24a^3 + 81b^3$ (6) $y^3 + \frac{1}{8y^3}$ (7) $a^3 + \frac{8}{a^3}$ (8) $1 + \frac{q^3}{125}$

Let's learn.

Factors of $a^3 - b^3$

$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 - 3ab(a - b)$

Now, $a^3 - b^3 - 3ab\,(a - b) = (a - b)^3$

$\therefore \quad a^3 - b^3 = (a - b)^3 + 3ab\,(a - b)$

$$= [(a - b)(a - b)^2 + 3ab\,(a - b)]$$

$$= (a - b)\,[(a - b)^2 + 3ab]$$

$$= (a - b)\,(a^2 - 2ab + b^2 + 3ab)$$

$$= (a - b)\,(a^2 + ab + b^2)$$

$\therefore$ $\boldsymbol{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$

Lets us solve some examples using the above formula for factorising the difference of two cubes.

Ex. (1) $x^3 - 8y^3 = x^3 - (2y)^3$

$\therefore\ x^3 - 8y^3 = x^3 - (2y)^3$

$= (x - 2y)\,(x^2 + 2xy + 4y^2)$

Ex. (2) $27p^3 - 125q^3 = (3p)^3 - (5q)^3 = (3p - 5q)\,(9p^2 + 15pq + 25q^2)$

Ex. (3) $54p^3 - 250q^3 = 2\,[27p^3 - 125q^3] = 2\,[(3p)^3 - (5q)^3]$

$= 2(3p - 5q)(9p^2 + 15pq + 25q^2)$

Ex. (4) $a^3 - \frac{1}{a^3} = \left(a - \frac{1}{a}\right)\left(a^2 + 1 + \frac{1}{a^2}\right)$

Ex. (5) Simplify : $(a - b)^3 - (a^3 - b^3)$

Solution : $(a - b)^3 - (a^3 - b^3) = a^3 - 3a^2b + 3ab^2 - b^3 - a^3 + b^3 = -\,3a^2b + 3ab^2$

Ex. (6) Simplify : $(2x + 3y)^3 - (2x - 3y)^3$

Solution : Using the formula $a^3 - b^3 = (a - b)\,(a^2 + ab + b^2)$

$\therefore (2x + 3y)^3 - (2x - 3y)^3$

$= [(2x+3y)-(2x-3y)][(2x+3y)^2+(2x+3y)(2x-3y)+(2x-3y)^2]$

$= [2x + 3y - 2x + 3y]\,[4x^2+12xy+9y^2 + 4x^2-9y^2+4x^2-12xy+9y^2]$

$= 6y\,(12x^2 + 9y^2) = 72x^2y + 54y^3$

Now I know.

(i) $a^3 + b^3 = (a + b)\,(a^2 - ab + b^2)$ (ii) $a^3 - b^3 = (a - b)\,(a^2 + ab + b^2)$

Practice Set 6.3

1. Factorise : (1) $y^3 - 27$ (2) $x^3 - 64y^3$ (3) $27m^3 - 216n^3$ (4) $125y^3 - 1$

(5) $8p^3 - \frac{27}{p^3}$ (6) $343a^3 - 512b^3$ (7) $64x^3 - 729y^3$ (8) $16a^3 - \frac{128}{b^3}$

2. Simplify : (1) $(x + y)^3 - (x - y)^3$ (2) $(3a + 5b)^3 - (3a - 5b)^3$

(3) $(a + b)^3 - a^3 - b^3$ (4) $p^3 - (p + 1)^3$

(5) $(3xy - 2ab)^3 - (3xy + 2ab)^3$

Let's learn.

Rational algebraic expressions

If A and B are two algebraic expressions then $\frac{A}{B}$ is called a rational algebraic expression. While simplifying a rational algebraic expression , we have to perform operations of addition, subtraction, multiplication and division. They are similar to those performed on rational numbers .

Note that, the denomintors or the divisors of algebraic expressions are non-zero.

Ex. (1) Simplify : $\frac{a^2+5a+6}{a^2-a-12}\times\frac{a-4}{a^2-4}$

Solution: $\frac{a^2+5a+6}{a^2-a-12}\times\frac{a-4}{a^2-4}$

$$=\frac{(a+3)(a+2)}{(a-4)(a+3)}\times\frac{(a-4)}{(a+2)(a-2)}$$

$$=\frac{1}{a-2}$$

Ex. (2) $\frac{7x^2+18x+8}{49x^2-16}\times\frac{14x-8}{x+2}$

Solution : $\frac{7x^2+18x+8}{49x^2-16}\times\frac{14x-8}{x+2}$

$$=\frac{(7x+4)(x+2)}{(7x+4)(7x-4)}\times\frac{2(7x-4)}{(x+2)}$$

$$=2$$

Ex. (3) Simplify : $\frac{x^2-9y^2}{x^3-27y^3}$

Solution: $\frac{x^2-9y^2}{x^3-27y^3}=\frac{(x+3y)(x-3y)}{(x-3y)(x^2+3xy+9y^2)}\quad=\frac{x+3y}{x^2+3xy+9y^2}$

Practice Set 6.4

1. Simplify :

(1) $\frac{m^2-n^2}{(m+n)^2}\times\frac{m^2+mn+n^2}{m^3-n^3}$ (2) $\frac{a^2+10a+21}{a^2+6a-7}\times\frac{a^2-1}{a+3}$ (3) $\frac{8x^3-27y^3}{4x^2-9y^2}$

(4) $\frac{x^2-5x-24}{(x+3)(x+8)}\times\frac{x^2-64}{(x-8)^2}$ (5) $\frac{3x^2-x-2}{x^2-7x+12}\div\frac{3x^2-7x-6}{x^2-4}$ (6) $\frac{4x^2-11x+6}{16x^2-9}$

(7) $\frac{a^3-27}{5a^2-16a+3}\div\frac{a^2+3a+9}{25a^2-1}$ (8) $\frac{1-2x+x^2}{1-x^3}\times\frac{1+x+x^2}{1+x}$

Answers

Practice Set 6.1

1. (1) $(x + 6)(x + 3)$ (2) $(x - 9)(x - 1)$ (3) $(y + 12)(y + 12)$

(4) $5(y + 2)(y - 1)$ (5) $(p - 7)(p + 5)$ (6) $(p + 4)(p - 11)$

(7) $(m - 15)(m - 8)$ (8) $(m - 20)(m - 5)$ (9) $(x + 3)(3x + 5)$

(10) $(x + 5)(2x - 9)$ (11) $2(5x - 4)(2x - 1)$ (12) $(11x - 3)(4x + 1)$

Practice Set 6.2

1. (1) $(x + 4y)(x^2 - 4xy + 16y^2)$ (2) $(5p + q)(25p^2 - 5pq + q^2)$

(3) $(5k + 3m)(25k^2 - 15km + 9m^2)$ (4) $2(l + 6m)(l^2 - 6lm + 36m^2)$

(5) $3(2a + 3b)(4a^2 - 6ab + 9b^2)$ (6) $\left(y+\frac{1}{2y}\right)\left(y^2-\frac{1}{2}+\frac{1}{4y^2}\right)$

(7) $\left(a+\frac{2}{a}\right)\left(a^2-2+\frac{4}{a^2}\right)$ (8) $\left(1+\frac{q}{5}\right)\left(1-\frac{q}{5}+\frac{q^2}{25}\right)$

Practice Set 6.3

1. (1) $(y - 3)(y^2 + 3y + 9)$ (2) $(x - 4y)(x^2 + 4xy + 16y^2)$

(3) $27(m - 2n)(m^2 + 2mn + 4n^2)$ (4) $(5y - 1)(25y^2 + 5y + 1)$

(5) $\left(2p-\frac{3}{p}\right)\left(4p^2+6+\frac{9}{p^2}\right)$ (6) $(7a - 8b)(49a^2 + 56ab + 64b^2)$

(7) $(4x - 9y)(16x^2 + 36xy + 81y^2)$ (8) $16\left(a-\frac{2}{b}\right)\left(a^2+\frac{2a}{b}+\frac{4}{b^2}\right)$

2. (1) $6x^2y + 2y^3$ (2) $270a^2b + 250b^3$ (3) $3a^2b + 3ab^2$

(4) $-3p^2 - 3p - 1$ (5) $-108x^2y^2ab - 16a^3b^3$

Practice Set 6.4

1. (1) $\frac{1}{m+n}$ (2) $a + 1$ (3) $\frac{4x^2+6xy+9y^2}{2x+3y}$

(4) 1 (5) $\frac{(x-1)(x-2)(x+2)}{(x-3)^2(x-4)}$

(6) $\frac{x-2}{4x+3}$ (7) $5a + 1$ (8) $\frac{1-x}{1+x}$

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