8. CONTINUITY

Let's study.

- Continuity of a function at a point.
- Continuity of a function over an interval.

Let's recall.

- Different types of functions.
- Limits of Algebraic, Exponential, Logarithmic functions.
- Left hand and Right hand limits of functions.

Let's learn.

8.1 CONTINUOUS AND DISCONTINUOUS FUNCTIONS

The dictionary meaning of the word continuity **is the unbroken and consistent existence over a period of time.** The intuitive idea of continuity is manifested in the following examples.

(i) An unbroken road between two cities.

(ii) Flow of river water.

(iii) The story of a drama.

(iv) Railway tracks.

(v) Temperature of a city on a day changing with time.

The temperature of Pune rises from 14^0 C at night to 29^0 C in the afternoon, this change in the temperature is continuous and all the values between 14 and 29 are taken during 12 hours. An activity that takes place gradually, without interruption or abrupt change is called a continuous process. Similarly to ensure continuity of a function, there should not be any interruption or jump, or break in the graph of a function.

8.2 CONTINUITY AT A POINT

We are going to study continuity of functions of real variable. So the domain will be an interval in R. Before we consider a formal definition of continuity of a function at a point, let's consider various functions .

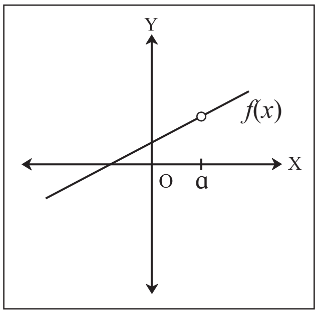


Figure 8.1

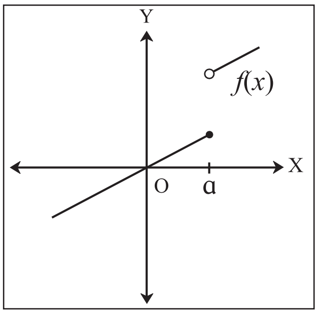


Figure 8.2

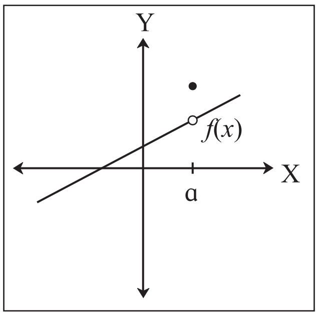


Figure 8.3

In Figure 8.1, we see that the graph of $f(x)$ has a hole at $x = a$. In fact, $f(a)$ is not defined for $x = a$. At the very least, for $f(x)$ to be continuous at $x = a$, we need $f(a)$ to be defined.

Condition 1 : If $f(x)$ is to be continuous at $x = a$ then $f(a)$ must be defined.

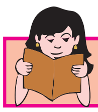
Now, let us see Figure 8.2. Although $f(a)$ is defined, the graph has a gap at $x = a$. In this example, the gap exists because $\lim_{x \to a} f(x)$ does not exist. Because, left hand limit at x = a is not equal to its right hand limit at x = a. So, we must add another condition for continuity at $x = a$, will be that $\lim_{x \to a} f(x)$ must exist. So the right hand limit and left hand limits are equal.

Condition 2 : If f(x) is to be continuous at $x = a$ then $\lim_{x \to a} f(x)$ must exist.

In Figure 8.3, The function in this figure satisfies both of our first two conditions, but is still there is a hole in the graph of the function. We must add a third condition that $\lim_{x \to a} f(x) = f(a)$.

Condition 3 : If f(x) is to be continuous at $x = a$ then $\lim_{x \to a} f(x) = f(a)$.

Now we put our list of conditions together and form the definition of continuity at a point.



Let's learn.

8.3 DEFINITION OF CONTINUITY

A function $f(x)$ is said to be continuous at point $x = a$ if the following three conditions are satisfied:

i. f is defined on an open interval containing a.

ii. $\lim_{x \to a} f(x)$ exists

iii. $\lim_{x \to a} f(x) = f(a)$.

The condition (iii) can be reformulated and the continuity of $f(x)$ at $x = a$, can be restated as follows :

A function $f(x)$ is said to be continuous at a point $x = a$ if it is defined in some neighborhood of 'a' and if $\lim_{h \to 0} [f(a+h) - f(a)] = 0$.

Discontinuous Function : A function $f(x)$ is not continuous at $x = c$, it is said to be discontinuous at $x = c$, 'c' is called the point of discontinuity.

Example 1. Consider $f(x) = |x|$ be defined on R.

$$f(x) = -x \quad \text{for } x < 0$$

$$= x \quad \text{for } x \geq 0 .$$

Let us discuss the continuity of $f(x)$ at $x = 0$

Consider, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x) = 0$$

$$\text{For} \quad x = 0, \ f(0) = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0$$

Hence $f(x)$ is continuous at $x = 0$.

Example 2 :

Consider $f(x) = x^2$. Let us discuss the continuity of f at $x = 2$.

$$f(x) = x^2$$

$$f(2) = 2^2 = 4$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x^2) = 2^2 = 4$$

$$\lim_{x \to 2} f(x) = f(2) = 4$$

Therefore the function $f(x)$ is continuous at x=2.

There are some functions, which are defined in two different ways on either side of a point. In such cases we have to consider the limits of function from left as well as from right of that point.

8.4 CONTINUITY FROM THE RIGHT AND FROM THE LEFT

A function $f(x)$ is said to be continuous from the right at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be continuous from the left at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.

Consider the following examples.

Example 1: Let us discuss the continuity of $f(x) = [x]$ in the interval [2, 4)

[Note : $[x]$ is the greatest integer function or floor function]

$$f(x) = [x], \quad \text{for } x \in [2, 4)$$

$$\text{that is } f(x) = 2, \quad \text{for } x \in [2, 3)$$

$$f(x) = 3, \quad \text{for } x \in [3, 4)$$

The graph of the function is as shown below,

Test of continuity at $x = 3$.

For $x = 3$, $f(3) = 3$

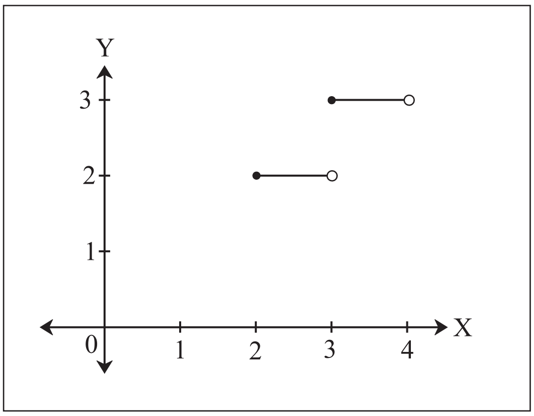


Figure 8.4

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} [x] = 2 \text{ and}$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} [x] = \lim_{x \to 3^+} (3) = 3$$

$$\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$$

Therefore $f(x)$ is discontinuous at $x = 3$.

Example 2 : Let us discuss the continuity of

$$f(x) = x^2 + 2 \text{ for } 0 \leq x \leq 2$$

$$= 5x - 4 \quad \text{for} \quad 2 < x \leq 3.5$$

For $x = 2$, $\quad f(2) = 2^2 + 2 = 6$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 + 2) = (2^2+2) = 6$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (5x - 4) = (10-4) = 6$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 6 \Rightarrow \lim_{x \to 2} f(x) = f(2)$$

Therefore $f(x)$ is continuous at $x = 2$.

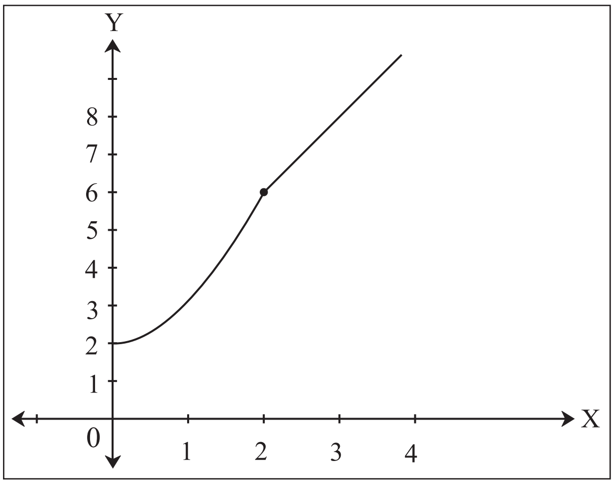


Figure 8.5

8.5 PROPERTIES OF CONTINUOUS FUNCTIONS:

If the functions f and g are continuous at $x = a$, then,

1. their sum ($f + g$) is continuous at $x = a$.
2. their difference ($f - g$) is continuous at $x = a$.
3. constant multiples that is $k.f$ for any $k \in R$ is continuous at $x = a$.
4. their product that is ($f.g$) is continuous at $x = a$.

5. their quotient that is $\frac{f}{g}$ where $g(a) \neq 0$ is continuous at $x = a$.

6. their composite function $f\,[g(x)]$, that is $fog\,(x)$ is continuous at $x = a$.

8.6 CONTINUITY OVER AN INTERVAL

So far we have explored the concept of continuity of a function at a point. Now we will extend the idea of continuity to an interval.

Let (a, b) be an open interval. If for every $x \in (a, b)$, f is continuous at x then we say that f is continuous on (a, b) .

Consider f on $[a, b)$ if f is continuous on (a, b) and f is continuous to the right of a, then f is continuous on $[a, b)$

Consider f on $(a, b]$ if f is continuous on

(a, b) and f is continuous to the left of b, then f is continuous on $(a, b]$

Consider a function f continuous on the open interval (a,b). If $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to b} f(x)$ exists, then we can extend the function to $[a,b]$ so that it is continuous on $[a,b]$.

Definition : A real valued function f is said to be continuous in an interval if it is continuous at every point of the interval.

8.7 CONTINUITY IN THE DOMAIN OF THE FUNCTION :

A real valued function $f : D \to R$ is said to be continuous if it is continuous at every point in the domain D of 'f'.

SOLVED EXAMPLES

Example 1 : Determine whether the function f is discontinuous for any real numbers in its domain.

where $f(x) = 3x + 1$, for $x < 2$

$= 7$, for $2 \leq x < 4$

$= x^2 - 8$ for $x \geq 4$.

Solution :

Let us check the conditions of continuity at $x = 2$.

For $x = 2$, $f(2) = 7$ (Given)

[Condition 1 is satisfied]

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (3x + 1) = 3(2) + 1 = 7.$$

and $\lim\limits_{x\to 2^+} f(x) = 7$,

So $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = 7 \Rightarrow \lim\limits_{x\to 2} f(x) = 7$

[Condition 2 is satisfied]

Also, $\lim\limits_{x\to 2} f(x) = 7 = f(2)$, $\lim\limits_{x\to 2} f(x) = f(2)$

[Condition 3 is satisfied]

Therefore $f(x)$ is continuous at $x = 2$.

Let us check the conditions of continuity at $x = 4$.

For $x = 4$, $f(4) = (4^2 - 8) = 8$

[Condition 1 is satisfied]

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (7) = 7 \text{ and}$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (x^2 - 8) = 4^2 - 8 = 8$$

$\lim\limits_{x\to 4^-} f(x) \neq \lim\limits_{x\to 4^+} f(x)$ So, $\lim\limits_{x\to 4} f(x)$ does not exist.

[Condition 2 is not satisfied]

Since one of the three conditions is not satisfied at $x = 4$, the function $f(x)$ is discontinuous at $x = 4$. Therefore the function f is continuous on it's domain, except at $x = 4$.

Example 2 : Test whether the function $f(x)$ is continuous at $x = -4$, where

$$f(x) = \frac{x^2 + 16x + 48}{x + 4} \quad \text{for } x \neq -4$$

$$= 8 \quad \text{for } x = -4.$$

Solution : For $x = -4$, $f(-4) = 8$ (defined)

$$\lim_{x\to -4} f(x) = \lim_{x\to -4} \left(\frac{x^2 + 16x + 48}{x + 4}\right)$$

$$= \lim_{x\to -4}\left(\frac{(x+4)(x+12)}{x+4}\right)$$

$$\lim_{x\to -4}(x+12) = -4+12 = 8$$

$$\therefore \lim_{x\to -4} f(x) \text{ exists}$$

$$\therefore \lim_{x\to -4} f(x) = f(-4) = 8$$

Therefore the function $f(x)$ is continuous at $x = -4$.

Example 3 : If $f(x) = \left(\frac{3x+2}{2-5x}\right)^{\frac{1}{x}}$ for $x \neq 0$,

is continuous at $x = 0$ then find $f(0)$.

Solution : Given that $f(x)$ is continuous at $x = 0$

$$\therefore f(0) = \lim_{x\to 0} f(x)$$

$$f(0) = \lim_{x\to 0}\left(\frac{3x+2}{2-5x}\right)^{\frac{1}{x}}$$

$$= \lim_{x\to 0}\left(\frac{2\left(1+\frac{3x}{2}\right)}{2\left(1-\frac{5x}{2}\right)}\right)^{\frac{1}{x}}$$

$$= \lim_{x\to 0}\left(\frac{\left(1+\frac{3x}{2}\right)^{\frac{1}{x}}}{\left(1-\frac{5x}{2}\right)^{\frac{1}{x}}}\right)$$

$$= \frac{\left[\lim_{x\to 0}\left(1+\frac{3x}{2}\right)^{\frac{2}{3x}}\right]^{\frac{3}{2}}}{\left[\lim_{x\to 0}\left(1-\frac{5x}{2}\right)^{\frac{-2}{5x}}\right]^{\frac{-5}{2}}}$$

$$= \frac{e^{\frac{3}{2}}}{e^{\frac{-5}{2}}}$$

$$= e^{\frac{3}{2}+\frac{5}{2}} = e^{\frac{8}{2}} = e^4 \quad \because \left[\lim_{x\to 0}(1+kx)^{\frac{1}{kx}} = e\right]$$

8.8 EXAMPLES OF CONTINUOUS FUNCTIONS WHEREVER THEY ARE DEFINED:

(1) Constant function is continuous at every point of R.

(2) Power functions with positive integer exponents are continuous at every point of R.

(3) Polynomial functions,

$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n$ are continuous at every point of R

(4) The exponential function a^x and logarithmic function $\log_b x$ (for any $x > 0$, base $b > 0$ and $b \neq 1$) are continuous for all $x \in R$

(5) Rational functions which are quotients of polynomials of the form $\frac{P(x)}{Q(x)}$ are continuous at every point , where $Q(x) \neq 0$.

(6) The n^{th} root functions, are continuous in their respective domains.

EXERCISE 8.1

1. Examine the continuity of

(i) $f(x) = x^3 + 2x^2 - x - 2$ at $x = -2$.

(ii) $f(x) = \frac{x^2-9}{x-3}$ on R

2. Examine whether the function is continuous at the points indicated against them.

(i) $f(x) = x^3 - 2x + 1$, for $x \leq 2$

$= 3x - 2$, for $x > 2$, at $x = 2$.

(ii) $f(x) = \dfrac{x^2+18x-19}{x-1}$ for $x \neq 1$

$= 20$ for $x = 1$, at $x = 1$

3. Test the continuity of the following functions at the points indicated against them.

(i) $f(x) = \dfrac{\sqrt{x-1}-(x-1)^{\frac{1}{3}}}{x-2}$ for $x \neq 2$

$= \dfrac{1}{5}$ for $x = 2$, at $x = 2$

(ii) $f(x) = \dfrac{x^3-8}{\sqrt{x+2}-\sqrt{3x-2}}$ for $x \neq 2$

$= -24$ for $x = 2$, at $x = 2$

(iii) $f(x) = 4x+1$, for $x \leq 3$

$= \dfrac{59-9x}{3}$, for $x > 3$ at $x = \dfrac{8}{3}$.

(iv) $f(x) = \dfrac{x^3-27}{x^2-9}$ for $0 \leq x < 3$

$= \dfrac{9}{2}$ for $3 \leq x \leq 6$

at $x = 3$

4) (i) If $f(x) = \dfrac{24^x-8^x-3^x+1}{12^x-4^x-3^x+1}$ for $x \neq 0$

$= k$, for $x = 0$

is continuous at $x = 0$, find k.

(ii) If $f(x) = \dfrac{5^x+5^{-x}-2}{x^2}$ for $x \neq 0$

$= k$ for $x = 0$

is continuous at $x = 0$, find k

(iii) For what values of a and b is the function

$f(x) = ax + 2b + 18$ for $x \leq 0$

$= x^2 + 3a - b$ for $0 < x \leq 2$

$= 8x - 2$ for $x > 2$,

continuous for every x ?

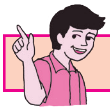
(iv) For what values of a and b is the function

$f(x) = \dfrac{x^2-4}{x-2}$ for $x < 2$

$= ax^2 - bx + 3$ for $2 \leq x < 3$

$= 2x - a + b$ for $x \geq 3$

continuous in its domain.



Let's remember!

Continuity at a point

A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied:

(1) $f(a)$ is defined, on an open interval containing a

(2) $\lim_{x \to a} f(x)$ exists, and

(3) $\lim_{x \to a} f(x) = f(a)$

Continuity from right : A function is continuous from right at a if $\lim_{x \to a^+} f(x) = f(a)$

Continuity from left : A function is continuous from left at b if $\lim_{x \to b^-} f(x) = f(b)$

Continuity over an interval :

Open Interval : A function is continuous over an open interval if it is continuous at every point in the interval.

Closed Interval : A function $f(x)$ is continuous over a closed interval of the form $[a,b]$ if it is continuous at every point in (a,b), and it is continuous from the right at a and continuous from the left at b

Discontinuity at a point : A function is discontinuous at a point if it is not continuous at that point.

MISCELLANEOUS EXERCISE - 8

(I) Discuss the continuity of the following functions at the point(s) or in the interval indicated against them.

1) $f(x) = 2x^2 - 2x + 5$ for $0 \le x < 2$

$= \dfrac{1-3x-x^2}{1-x}$ for $2 \le x < 4$

$= \dfrac{7-x^2}{x-5}$ for $4 \le x \le 7$ on its domain.

2) $f(x) = \dfrac{3^x+3^{-x}-2}{x^2}$ for $x \ne 0$.

$= (\log 3)^2$ for $x = 0$ at $x = 0$

3) $f(x) = \dfrac{5^x-e^x}{2x}$ for $x \ne 0$

$= \dfrac{1}{2}(\log 5-1)$ for $x = 0$ at $x = 0$.

4) $f(x) = \dfrac{\sqrt{x+3}-2}{x^3-1}$ for $x \ne 1$

$= 2$ for $x = 1$, at $x = 1$.

5) $f(x) = \dfrac{\log x - \log 3}{x-3}$ for $x \ne 3$

$= 3$ for $x = 3$, at $x = 3$.

(II) Find k if following functions are continuous at the points indicated against them.

(1) $f(x) = \left(\dfrac{5x-8}{8-3x}\right)^{\frac{3}{2x-4}}$ for $x \ne 2$

$= k$ for $x = 2$ at $x = 2$.

(2) $f(x) = \dfrac{45^x-9^x-5^x+1}{(k^x-1)(3^x-1)}$ for $x \ne 0$

$= \dfrac{2}{3}$ for $x = 0$, at $x = 0$

(3) $f(x) = (1+kx)^{\frac{1}{x}}$, for $x \ne 0$

$= e^{\frac{3}{2}}$, for $x = 0$, at $x = 0$

(III) Find a and b if following functions are continuous at the point indicated against them.

(1) $f(x) = x^2 + a$, for $x \ge 0$

$= 2\sqrt{x^2+1} + b$, for $x < 0$ and

$f(1) = 2$ is continuous at $x = 0$

(2) $f(x) = \dfrac{x^2-9}{x-3} + a$, for $x > 3$

$= 5$, $x = 3$

$= 2x^2 + 3x + b$, for $x < 3$

is continuous at $x = 3$

(3) $f(x) = \dfrac{32^x-1}{8^x-1} + a$, for $x > 0$

$= 2$, for $x = 0$

$= x + 5 - 2b$, for $x < 0$

is continuous at $x = 0$

ACTIVITIES

Activity 8.1:

If the following function is continuous at $x = 0$, find a and b.

$f(x) = x^2 + a$, for $x > 0$

$= 2\sqrt{x^2+1} + b$, for $x < 0$

$= 2$, for $x = 0$

Solution : Given

$f(x) = x^2 + a$, for $x > 0$

$= 2\sqrt{x^2+1} + b$, for $x < 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^2 + a)$

$= (\square^2 + a)$

$\lim_{x\to 0^+} f(x) = \square$ .... (1)

$f(0) = 2$ .... (2)

Since $f(x)$ is continuous at $x = 0$

$\therefore \lim_{x\to 0^+} f(x) = f(0)$

$\therefore a = \square$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2\sqrt{x^2+1} + b)$

$= 2\sqrt{\square + 1} + b$

$\lim_{x\to 0^-} f(x) = \square + b$ .... (3)

Since f (x) is continuous at $x = 0$

$\lim_{x\to 0^-} f(x) = f(0)$

$\therefore \square + b = 2$

$\therefore b = \square$

Activity 8.2:

If $f(x) = \dfrac{x^2-4}{x-2}$, for $x \neq 2$ is continuous at $x = 2$ then find $f(2)$.

Solution : Given function,

$$f(x) = \frac{x^2-4}{x-2}$$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2}$$

$$= \lim_{x\to 2} \frac{x^2 - \square}{x - \square}$$

$$= \lim_{x\to 2} \frac{(x+\square)(x-\square)}{(x-\square)}$$

$$= \lim_{x\to 2} (x+\square)$$

$$= \square + \square$$

$\lim_{x\to 2} f(x) = \square$

Since $f(x)$ is continuous at $x = 2$

$\lim_{x\to 2} f(x) = f(2)$

$\lim_{x\to 2} f(x) = \square$

$\therefore f(2) = \square$

Activity 8.3:

Determine whether the function 'f' is continuous on its domain.

$$f(x) = 3x + 1, \quad x < 2$$
$$= 7, \quad 2 \leq x < 4$$
$$= x^2 - 8, \quad x \geq 4$$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \square$

$= \square$

Now $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \square = \square$

But $f(x) = \square$ at $x = 2$

$\therefore$ The function is $\square$ at $x = 2$

Also $\lim_{x\to 4^-} f(x) = \square$

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \square = \square$

$\therefore$ Given function is $\square$ at $x = 4$

$\therefore$ the function is continuous at $x = \square$ and discontinuous at $x = \square$ in the domain of $f(x)$.

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