3. COMPLEX NUMBERS

Let's study.

- Complex number
- Algebra of complex number
- Solution of Quadratic equation
- Cube roots of unity

Let's recall.

- Algebra of real numbers
- Solution of linear and quadratic equations
- Representation of a real number on the number line

Let's learn.

- Representation of point in a plane

Introduction:

Consider the equation $x^2 + 1 = 0$. This equation has no solution in the set of real numbers because there is no real number whose square is negative. To extend the set of real numbers to a larger set, which would include such solutions.

We introduce the symbol i such that $i = \sqrt{-1}$ and $i^2 = -1$.

Symbol i is called as an **imaginary unit.**

Swiss mathematician Euler (1707-1783) was the first mathematician to introduce the symbol i with $i^2 = -1$.

3.1 IMAGINARY NUMBER :

A number of the form ki, where $k \in$ R, $k \neq 0$ and $i = \sqrt{-1}$ is called an imaginary number.

For example

$\sqrt{-25} = 5i, 2i, \frac{2}{7}i, -11i, \sqrt{-4}$ etc.

Let's Note.

The number i satisfies following properties,

i) $i \times 0 = 0$

ii) If $a \in$ R, then $\sqrt{-a^2} = \sqrt{i^2 a^2} = \pm ia$

iii) If $a, b \in$ R, and $ai = bi$ then $a = b$

3.2 COMPLEX NUMBER :

Definition : A number of the form $a+ib$, where $a, b \in$ R and $i = \sqrt{-1}$ is called a complex number and it is denoted by z.

$\therefore \quad z = a+ib = a + bi$

Here a is called the real part of z and is denoted by **Re(z) or R(z)**

'b' is called the imaginary part of z and is denoted by **Im(z) or I(z)**

The set of complex numbers is denoted by C

$\therefore$ C = $\{a+ib \mid a, b \in$ R, and $i = \sqrt{-1}\}$

For example

z	a+ib	Re(z)	Im(z)
$2+4i$	$2+4i$	2	4
$5i$	$0+5i$	0	5
$3-4i$	$3-4i$	3	-4
$5+\sqrt{-16}$	$5+4i$	5	4
$2+\sqrt{-5}$	$2+\sqrt{5}\,i$	2	$\sqrt{5}$
$7+\sqrt{3}$	$(7+\sqrt{3})+0i$	$(7+\sqrt{3})$	0

Let's Note.

1) A complex number whose real part is zero is called a imaginary number. Such a number is of the form $z = 0 + ib = ib = bi$

2) A complex number whose imaginary part is zero is a real number.
   $z = a + 0i = a$, for every real number.

3) A complex number whose both real and imaginary parts are zero is the zero complex number.
   $0 = 0 + 0i$

4) The set R of real numbers is a subset of the set C of complex numbers.

3.2.1 Conjugate of a Complex Number :

Definition : The conjugate of a complex number $z=a+ib$ is defined as $a - ib$ and is denoted by $\overline{z}$

For example

z	$\overline{z}$
$3 + 4i$	$3 - 4i$
$7i - 2$	$-7i - 2$
3	3
$5i$	$-5i$
$2 + \sqrt{3}$	$2 + \sqrt{3}$
$7 + \sqrt{-5}$	$7 - \sqrt{5}\, i$

Properties of $\overline{z}$

1) $(\overline{\overline{z}}) = z$

2) If $z = \overline{z}$, then z is real.

3) If $z = -\overline{z}$, then z is imaginary.

Now we define the four fundamental operations of addition, subtraction, multiplication and division of complex numbers.

3.3 ALGEBRA OF COMPLEX NUMBERS :

3.3.1 Equality of two Complex Numbers :

Definition : Two complex numbers $z_1 = a+ib$ and $z_2 = c + id$ are said to be equal if their real and imaginary parts are equal, i.e. $a = c$ and $b = d$.

For example,

i) If $x + iy = 4 + 3i$ then $x = 4$ and $y = 3$

1) Addition :

let $z_1 = a+ib$ and $z_2 = c+id$

then $z_1 + z_2 = a+ib + c+id$

$= (a+c) + (b+d)\, i$

Hence, $\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2)$

and $\text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$

Ex. 1) $(2 + 3i) + (4 + 3i) = (2+4) + (3+3)i$

$= 6 + 6i$

2) $(-2 + 5i) + (7 + 3i) + (6 - 4i)$

$= [(-2) + 7 + 6] + [5 + 3 + (-4)]i$

$= 11 + 4i$

Properties of addition : If z_1, z_2, z_3 are complex numbers then

i) $z_1 + z_2 = z_2 + z_1$

ii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

iii) $z_1 + 0 = 0 + z_1 = z_1$

iv) $z + \overline{z} = 2\text{Re}(z)$

v) $\left(\overline{z_1 + z_2}\right) = \overline{z_1} + \overline{z_2}$

2) Scalar Multiplication :

If $z = a+ib$ is any complex number, then for every real number k, we define $kz = ka + i(kb)$

Ex. 1) If $z = 7 + 3i$ then

$5z = 5(7 + 3i) = 35 + 15i$

3) Subtraction :

Let $z_1 = a+ib$, $z_2 = c+id$ then

$z_1 - z_2 = z_1 + (-z_2) = (a+ib) + (-c-id)$

[Here $-z_2 = -1(z_2)$]

$= (a-c) + i\ (b-d)$

Hence,

$\text{Re}(z_1 - z_2) = \text{Re}(z_1) - \text{Re}(z_1)$

$\text{Im}(z_1 - z_2) = \text{Im}(z_1) - \text{Im}(z_2)$

Ex.1) $z_1 = 4+3i,\ z_2 = 2+i$

$\therefore z_1 - z_2 = (4+3i) - (2+i)$

$= (4-2) + (3-1)\, i$

$= 2 + 2i$

Ex. 2) $z_1 = 7+i,\ z_2 = 4i,\ z_3 = -3+2i$

then $2z_1 - (5z_2 + 2z_3)$

$= 2(7+i) - [5(4i) + 2(-3+2i)]$

$= 14 + 2i - [20i - 6 + 4i]$

$= 14 + 2i - [-6 + 24i]$

$= 14 + 2i + 6 - 24i$

$= 20 - 22i$

4) Multiplication :

Let $z_1 = a+ib$ and $z_2 = c+id$. We denote multiplication of z_1 and z_2 as $z_1.z_2$ and is given by

$z_1.z_2 = (a+ib)(c+id) = a(c+id)+ib(c+id)$

$= ac + adi + bci + i^2 bd$

$= ac + (ad+bc)i - bd\ (\because i^2 = -1)$

$z_1.z_2 = (ac-bd) + (ad+bc)i$

Ex. $z_1 = 2+3i,\ z_2 = 3-2i$

$\therefore z_1.z_2 = (2+3i)(3-2i) = 2(3-2i) + 3i(3-2i)$

$= 6 - 4i + 9i - 6i^2$

$= 6 - 4i + 9i + 6 \qquad (\because i^2 = -1)$

$= 12 + 5i$

Properties of Multiplication : If $z_1,\ z_2,\ z_3$ are complex numbers, then

i) $z_1.z_2 = z_2.z_1$

ii) $(z_1.z_2).z_3 = z_1.(z_2.z_3)$

iii) $(z_1.1) = 1.z_1 = z_1$

iv) $z.\bar{z}$ is real number.

v) $\left(\overline{z_1.z_2}\right) = \bar{z}_1 . \bar{z}_2$ (Verify)

Let's Note.

If $z = a+ib$ then $z.\bar{z} = a^2 + b^2$

We have,

$i = \sqrt{-1}$, , $i^2 = -1$,

$i^3 = -i$, $i^4 = 1$

Powers of *i* :

In general,

$i^{4n} = 1$, $i^{4n+1} = i$,

$i^{4n+2} = -1$, $i^{4n+3} = -i$ where $n \in \text{N}$

5) Division :

Let $z_1 = a+ib$ and $z_2 = c+id$ be any two complex numbers such that $z_2 \neq 0$

Now,

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} \quad \text{where } z_2 \neq 0 \text{ i.e. } c+id \neq 0$$

The division can be carried out by multiplying and dividing $\frac{z_1}{z_2}$ by conjugate of $c+id$.

SOLVED EXAMPLES

Ex. 1) If $z_1 = 3+2i$, and $z_2 = 1+i$,

then write $\frac{z_1}{z_2}$ = in the form $a + ib$

Solution : $\frac{3+2i}{1+i} = \frac{3+2i}{1+i} \times \frac{1-i}{1-i}$

$$= \frac{3-3i+2i-2i^2}{(1)^2-(i)^2}$$

$$= \frac{3-3i+2i+2}{1+1} \qquad \because (i^2=-1)$$

$$= \frac{5-i}{2}$$

$$= \frac{5}{2} - \frac{1}{2}i$$

Ex. 2) Express $(1+2i)(-2+i)$ in the form of $a+ib$ where $a, b \in R$.

Solution : $(1+2i)(-2+i) = -2+i-4i+2i^2$

$= -2-3i-2$

$= -4-3i$

Ex. 3) Write $(1+2i)(1+3i)(2+i)^{-1}$ in the form $a+ib$

Solution :

$$(1+2i)(1+3i)(2+i)^{-1} = \frac{(1+2i)(1+3i)}{2+i}$$

$$= \frac{1+3i+2i+6i^2}{2+i}$$

$$= \frac{-5+5i}{2+i} \times \frac{2-i}{2-i}$$

$$= \frac{-10+5i+10i-5i^2}{4-i^2}$$

$$= \frac{-5+15i}{4+1} \qquad \because (i^2=-1)$$

$$= \frac{-5+15i}{5}$$

$= -1 + 3i$

Ex. 4) Express $\frac{1}{i} + \frac{2}{i^2} + \frac{3}{i^3} + \frac{5}{i^4}$ in the form of $a+ib$

Solution : We know that, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

$$\therefore \frac{1}{i} + \frac{2}{i^2} + \frac{3}{i^3} + \frac{5}{i^4}$$

$$= \frac{1}{i} + \frac{2}{-1} + \frac{3}{-i} + \frac{5}{1}$$

$$= \frac{1}{i} - \frac{3}{i} - 2 + 5$$

$$= \frac{-2}{i} + 3 = 2i + 3 \text{ (verify!)}$$

Ex. 5) If a and b are real and $(i^4+3i)a + (i-1)b + 5i^3 = 0$, find a and b.

Solution : $(i^4+3i)a + (i-1)b + 5i^3 = 0+0i$

i.e. $(1+3i)a + (i-1)b - 5i = 0+0i$

$\therefore a + 3ai + bi - b - 5i = 0+0i$

i.e. $(a-b) + (3a+b-5)i = 0+0i$

Equating real and imaginary parts, we get

$a-b = 0$ and $3a+b-5 = 0$

$\therefore a=b$ and $3a+b = 5$

$\therefore 3a+a = 5$

i.e. $4a = 5$

or $a = \frac{5}{4}$

$\therefore a = b = \frac{5}{4}$

Ex. 6) If $x + 2i + 15i^6y = 7x + i^3(y+4)$ find $x + y$, given that $x, y \in R$.

Solution :

$x + 2i + 15i^6y = 7x + i^3(y+4)$

$\therefore x + 2i - 15y = 7x - (y+4)i$

$\therefore x - 15y + 2i = 7x - (y+4)i$

Equating real and imaginary parts, we get

$x - 15y = 7x$ and $2 = -(y+4)$

$\therefore -6x - 15y = 0$ .... (i) $y+6 = 0$ ....(ii)

$\therefore y = -6, \; x = 15$

$\therefore x + y = 15 - 6 = 9$

Ex. 7) Find the value of $x^3 - x^2 + 2x + 4$ when $x = 1 + \sqrt{3}\ i$.

Solution : Since $x = 1 + \sqrt{3}\ i$

$\therefore (x-1) = \sqrt{3}\ i$

squaring both sides, we get

$(x-1)^2 = (\sqrt{3}\ i)^2$

$\therefore x^2 - 2x + 1 = 3i^2$

i.e. $x^2 - 2x + 1 = -3$

$\therefore x^2 - 2x + 4 = 0$

Now, consider

$x^3 - x^2 + 2x + 4 = x(x^2 - x + 2) + 4$

$= x(x^2 - 2x + 4 + x - 2) + 4$

$= x(0 + x - 2) + 4$

$= x^2 - 2x + 4$

$= 0$

Ex. 8) Show that $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^3 = i.$

Solution :

$$\text{L.H.S.} = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^3 = \left(\frac{\sqrt{3}+i}{2}\right)^3$$

$$= \frac{\left(\sqrt{3}\right)^3 + 3\left(\sqrt{3}\right)^2 i + 3\left(\sqrt{3}\right)i^2 + (i)^3}{(2)^3}$$

$$= \frac{3\sqrt{3} + 9i - 3\sqrt{3} - i}{8}$$

$$= \frac{8i}{8}$$

$= i$

= R.H.S.

Ex. 9) If $x = -5 + 2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 64$.

Solution : $x = -5 + 2\sqrt{-4}$

$\therefore x = -5 + 4i$

$\therefore x + 5 = 4i$

On squaring both sides

$(x+5)^2 = (4i)^2$

$\therefore x^2 + 10x + 25 = -16$

$\therefore x^2 + 10x + 41 = 0$

$$\begin{array}{r|l} & x^2 - x + 4 \\ \hline x^2 + 10x + 41 & x^4 + 9x^3 + 35x^2 - x + 64 \\ & x^4 + 10x^3 + 41x^2 \\ \hline & -x^3 - 6x^2 - x + 64 \\ & -x^3 - 10x^2 - 41x \\ \hline & 4x^2 + 40x + 64 \\ & 4x^2 + 40x + 164 \\ \hline & -100 \end{array}$$

$\therefore x^4 + 9x^3 + 35x^2 - x + 64$

$= (x^2 + 10x + 41)(x^2 - x + 4) - 100$

$= 0 \times (x^2 - x + 4) - 100$

$= -100$

EXERCISE 3.1

1) Write the conjugates of the following complex numbers

i) $3+i$ ii) $3-i$ iii) $-\sqrt{5} - \sqrt{7}\ i$

iv) $-\sqrt{-5}$ v) $5i$ vi) $\sqrt{5} - i$

vii) $\sqrt{2} + \sqrt{3}\ i$

2) Express the following in the form of a+ib, a, b∈R $i = \sqrt{-1}$. State the value of a and b.

i) $(1+2i)(-2+i)$ ii) $\frac{i(4+3i)}{(1-i)}$

iii) $\frac{(2+i)}{(3-i)(1+2i)}$ iv) $\frac{3+2i}{2-5i} + \frac{3-2i}{2+5i}$

v) $\frac{2+\sqrt{-3}}{4+\sqrt{-3}}$ vi) $(2+3i)(2-3i)$

vii) $\dfrac{4i^8 - 3i^9 + 3}{3i^{11} - 4i^{10} - 2}$

3) Show that $(-1+\sqrt{3}\,i)^3$ is a real number.

4) Evaluate the following :

i) i^{35} ii) i^{888} iii) i^{93} iv) i^{116}

v) i^{403} vi) $\dfrac{1}{i^{58}}$ vii) $i^{30}+i^{40}+i^{50}+i^{60}$

5) Show that $1 + i^{10}+i^{20}+i^{30}$ is a real number.

6) Find the value of

i) $i^{49}+i^{68}+i^{89}+i^{110}$

ii) $i+i^2+i^3+i^4$

7) Find the value of $1 + i^2+i^4+i^6+i^8+...+i^{20}$

8) Find the value of x and y which satisfy the following equations ($x, y\in$R)

i) $(x+2y) + (2x-3y)\,i + 4i = 5$

ii) $\dfrac{x+1}{1+i} + \dfrac{y-1}{1-i} = i$

9) Find the value of

i) $x^3 - x^2+x+46$, if $x = 2+3i$.

ii) $2x^3 - 11x^2+44x+27$, if $x = \dfrac{25}{3-4i}$.

3.4 Square root of a complex number :

Consider $z = x+iy$ be any complex number

Let $\sqrt{x+iy} = a+ib,\ a,\ b\in$R

On squaring both the sides, we get

$x+iy = (a+ib)^2$

$x+iy = (a^2-b^2) + (2ab)\,i$

Equating real and imaginary parts, we get

$x = (a^2-b^2)$ and $y = 2ab$

Solving the equations simultaneously, we can get the values of a and b.

SOLVED EXAMPLES

1) Find the square root of $6+8i$

Let the square root of $6+8i$ be $a+ib$, ($a,\ b\in$R)

$\therefore \sqrt{6+8i} = a+ib,\ a,\ b\in$R

On squaring both the sides, we get

$6+8i = (a+ib)^2$

$\therefore 6+8i = a^2-b^2+2abi$

Equating real and imaginary parts, we have

$6 = a^2-b^2$ .... (1)

$8 = 2ab$ .... (2)

$\therefore a = \dfrac{4}{b}$

$\therefore 6 = \left(\dfrac{4}{b}\right)^2 - b^2$

i.e. $6 = \dfrac{16}{b^2} - b^2$

$\therefore b^4+6b^2-16 = 0$

put $b^2 = m$

$\therefore m^2+6m-16 = 0$

$\therefore (m+8)(m-2) = 0$

$\therefore m = -8$ or $m = 2$

i.e. $b^2 = -8$ or $b^2 = 2$

but b$\in$R $\therefore b^2 \neq -8$

$\therefore b^2 = 2$ $\therefore b = \pm\sqrt{2}$

when $b = \sqrt{2}$, $a = 2\sqrt{2}$

$\therefore$ Square root of

$6+8i = 2\sqrt{2} + \sqrt{2}\,i = \sqrt{2}\,(2+i)$

when $b = -\sqrt{2}$, $a = -2\sqrt{2}$

$\therefore$ Square root of

$6+8i = -2\sqrt{2} - \sqrt{2}\,i = -\sqrt{2}\,(2+i)$

$\therefore \sqrt{6+8i} = \pm\sqrt{2}\left(2+i\right)$

Ex. 2 : Find the square root of $2i$

Solution :

Let $\sqrt{2i} = a+ib \qquad a, b \in R$

On squaring both the sides, we have

$2i = (a+ib)^2$

$\therefore 0+2i = a^2-b^2+2iab$

Equating real and imaginary parts, we have

$a^2-b^2 = 0, \quad 2ab = 2, \quad ab = 1$

As $(a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2$

$(a^2+b^2)^2 = 0^2 + 2^2$

$(a^2+b^2)^2 = 2^2$

$\therefore a^2+b^2 = 2$

Solving $a^2+b^2 = 2$ and $a^2-b^2 = 0$ we get

$2a^2 = 2$

$a^2 = 1$

$a = \pm 1$

But $b = \dfrac{2}{2a} = \dfrac{1}{a} = \dfrac{1}{\pm 1} = \pm 1$

$\therefore \sqrt{2i} = 1+i$ or $-1-i$

i.e. $\sqrt{2i} = \pm(1+i)$

3.5 Solution of a Quadratic Equation in complex number system :

Let the given equation be $ax^2 + bx + c = 0$ where $a, b, c \in R$ and $a \neq 0$

$\therefore$ the solution of this quadratic equation is given by

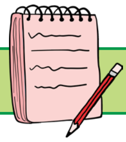
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Hence, the roots of the equation $ax^2+bx+c = 0$

are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$

The expression $(b^2-4ac) = D$ is called the discriminant.

If $D < 0$ then the roots of the given quadratic equation are not real in nature, that is the roots of such equation are complex numbers.



Let's Note.

If $p + iq$ is a root of equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ and $a \neq 0$ then $p - iq$ is also a root of the given equation. That is complex roots occurs in conjugate pairs.

SOLVED EXAMPLES

Ex. 1 : Solve $x^2 + x + 1 = 0$

Solution : Given equation is $x^2 + x + 1 = 0$

where $a = 1, b = 1, c = 1$

So, the given equation has complex roots.

These roots are given by

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

$\therefore$ Roots are $\dfrac{-1 \pm \sqrt{3}i}{2}$ and $\dfrac{-1-\sqrt{3}i}{2}$

Ex. 2 : Solve the following quadratic equation $x^2 - 4x + 13 = 0$

Solution : The given quadratic equation is

$x^2 - 4x + 13 = 0$

$x^2 - 4x + 4 + 9 = 0$

$\therefore \quad (x-2)^2 + 3^2 = 0$

$(x-2)^2 = -3^2$

$(x-2)^2 = +3^2.i^2$

taking square root we get,

$(x-2) = \pm 3i$

$\therefore \quad x = 2 \pm 3i$

$\therefore \quad x = 2 + 3i$ or $x = 2 - 3i$

Solution set = $\{2 + 3i, 2 - 3i\}$

Ex. 3 : Solve $x^2 + 4ix - 5 = 0$; where $i = \sqrt{-1}$

Solution : Given quadratic equation is

$$x^2 + 4ix - 5 = 0$$

Compairing with $ax^2 + bx + c = 0$

$$a = 1, \quad b = 4i, \quad c = -5$$

Consider $b^2 - 4ac = (4i)^2 - 4(1)(-5)$

$= 16i^2 + 20$

$= -16 + 20 \quad (\therefore i^2 = -1)$

$= 4$

The roots of quadratic equation are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4i \pm \sqrt{4}}{2(1)}$$

$$= \frac{-4i \pm 2}{2}$$

$$x = -2i \pm 1$$

Solution set = $\{-2i + 1, -2i - 1\}$

EXERCISE 3.2

1) Find the square root of the following complex numbers

i) $-8-6i$ ii) $7+24i$ iii) $1+4\sqrt{3}\,i$

iv) $3+2\sqrt{10}\,i$ v) $2(1-\sqrt{3}\,i)$

2) Solve the following quadratic equations.

i) $8x^2 + 2x + 1 = 0$

ii) $2x^2 - \sqrt{3}\,x + 1 = 0$

iii) $3x^2 - 7x + 5 = 0$

iv) $x^2 - 4x + 13 = 0$

3) Solve the following quadratic equations.

i) $x^2 + 3ix + 10 = 0$

ii) $2x^2 + 3ix + 2 = 0$

iii) $x^2 + 4ix - 4 = 0$

iv) $ix^2 - 4x - 4i = 0$

4) Solve the following quadratic equations.

i) $x^2 - (2+i)x - (1-7i) = 0$

ii) $x^2 - (3\sqrt{2} + 2i)x + 6\sqrt{2}\,i = 0$

iii) $x^2 - (5-i)x + (18+i) = 0$

iv) $(2 + i)x^2 - (5-i)x + 2(1-i) = 0$

3.6 Cube roots of unity :

Number 1 is often called unity. Let x be a cube root of unity.

$\therefore x^3 = 1$

$\therefore x^3 - 1 = 0$

$\therefore (x-1)(x^2 + x + 1) = 0$

$\therefore x - 1 = 0$ or $x^2 + x + 1 = 0$

$$\therefore x = 1 \text{ or } x = \frac{-1 \pm \sqrt{(1)^2 - 4\times1\times1}}{2\times1}$$

$$\therefore x = 1 \text{ or } x = \frac{-1 \pm \sqrt{-3}}{2}$$

$$\therefore x = 1 \text{ or } x = \frac{-1 \pm i\sqrt{3}}{2}$$

$\therefore$ Cube roots of unity are, 1, $\frac{-1 + i\sqrt{3}}{2}$, $\frac{-1 - i\sqrt{3}}{2}$

Among the three cube roots of unity, one is real and other two roots are complex conjugates of each other.

Now consider

$$\left(\frac{-1+\sqrt{3}\,i}{2}\right)^2$$

$$= \frac{1}{4}\left[(-1)^2 + 2\times(-1)i\sqrt{3} + \left(i\sqrt{3}\right)^2\right]$$

$$= \frac{1}{4}\,(1 - 2i\sqrt{3} - 3)$$

$$= \frac{1}{4}\,(-2 - 2i\sqrt{3})$$

$$= \frac{-1 - i\sqrt{3}}{2}$$

Similarly it can be verified that $\left(\frac{-1 - i\sqrt{3}}{2}\right)^2$

$\dfrac{-1+i\sqrt{3}}{2}$

Thus complex roots of unit are squares of each other. Thus cube roots of unity are given by

$$1, \frac{-1+i\sqrt{3}}{2}, \left(\frac{-1-i\sqrt{3}}{2}\right)^2$$

Let $\dfrac{-1+i\sqrt{3}}{2} = w$, then $\dfrac{-1-i\sqrt{3}}{2} = w^2$

Hence, cube roots of unity are 1, w, w^2 or 1, w, $\overline{w}$

where $w = \dfrac{-1+i\sqrt{3}}{2}$ and $w^2 = \dfrac{-1-i\sqrt{3}}{2}$

w is complex cube root of 1.

$\therefore\ w^3 = 1 \quad \therefore\ w^3 - 1 = 0$

i.e. $(w-1)(w^2+w+1) = 0$

$\therefore\ w=1$ or $w^2+w+1 = 0$

but $w \neq 1$

$\therefore\ w^2+w+1 = 0$

Properties of 1, w, w^2

i) $w^2 = \dfrac{1}{w}$ and $\dfrac{1}{w^2} = w$

ii) $w^3 = 1$ so $w^{3n} = 1$

iii) $w^4 = w^3 . w = w$ so $w^{3n+1} = w$

iv) $w^5 = w^2 . w^3 = w^2 . 1 = w^2$

So $w^{3n+2} = w^2$

v) $\overline{w} = w^2$

vi) $(\overline{w})^2 = w$

SOLVED EXAMPLES

Ex. 1 : If w is a complex cube root of unity, then prove that

i) $\dfrac{1}{w} + \dfrac{1}{w^2} = -1$

ii) $(1+w^2)^3 = -1$

iii) $(1-w+w^2)^3 = -8$

Solution : Given, w is a complex cube root of unity.

$\therefore\ w^3 = 1$ Also $w^2+w+1 = 0$

$\therefore\ w^2+1 = -w$ and $w+1 = -w^2$

i) $\dfrac{1}{w} + \dfrac{1}{w^2} = \dfrac{w+1}{w^2} = \dfrac{-w^2}{w^2} = -1$

ii) $(1+w^2)^3 = (-w)^3 = -w^3 = -1$

iii) $(1-w+w^2)^3 = (1+w^2-w)^3$

$= (-w-w)^3 \quad (\because\ 1+w^2 = -w)$

$= (-2w)^3$

$= -8w^3$

$= -8 \times 1$

$= -8$

Ex. 2 : If w is a complex cube root of unity, then show that

$(1-w)(1-w^2)(1-w^4)(1-w^5) = 9$

Solution : $(1-w)(1-w^2)(1-w^4)(1-w^5)$

$= (1-w)(1-w^2)(1-w^3.w)(1-w^3.w^2)$

$= (1-w)(1-w^2)(1-w)(1-w^2)$

$= (1-w)^2(1-w^2)^2$

$= [(1-w)(1-w^2)]^2$

$= (1-w^2-w+w^3)^2$

$= [1-(w^2+w)+1]^2$

$= [1-(-1)+1]^2$

$= (1+1+1)^2$

$= (3)^2$

$= 9$

Ex. 3 : Prove that

$1+w^n+w^{2n} = 3$, if n is a multiple of 3

$1+w^n+w^{2n} = 0$, if n is not multiple of 3, $n \in N$

Solution : If n is a multiple of 3 then n=3k and if n is not a multiple of 3 then n = 3k+1 or n = 3k+2, where k $\in$ N

Case 1: If n is multiple of 3

then $1+w^n+w^{2n} = 1+w^{3k}+w^{2\times 3k}$

$= 1+(w^3)^k+(w^3)^{2k}$

$= 1+(1)^k+(1)^{2k}$

$=1+1+1$

$= 3$

Case 2: If n = 3k + 1

then $1+w^n+w^{2n} = 1+(w)^{3k+1}+(w^2)^{3k+1}$

$= 1+(w^3)^k.w +(w^3)^{2k}.w^2$

$= 1+(1)^k.w +(1)^{2k}.w^2$

$=1+w+w^2$

$= 0$

Similarly by putting n = 3k+2, we have,

$1+w^n+w^{2n}=0$. Hence the results.

EXERCISE 3.3

1) If w is a complex cube root of unity, show that

i) $(2-w)(2-w^2) =7$

ii) $(2+w+w^2)^3-(1-3w+w^2)^3 = 65$

iii) $\dfrac{(a+bw+cw^2)}{c+aw+bw^2} = w^2$

2) If w is a complex cube root of unity, find the value of

i) $w + \dfrac{1}{w}$ ii) $w^2+w^3+w^4$ iii) $(1+w^2)^3$

iv) $(1-w-w^2)^3 + (1-w+w^2)^3$

v) $(1+w)(1+w^2)(1+w^4)(1+w^8)$

3) If α and β are the complex cube roots of unity, show that

$\alpha^2+\beta^2+\alpha\beta = 0$

4) If $x=a+b$, $y=\alpha a+\beta b$, and $z=a\beta+b\alpha$ where α and β are the complex cube-roots of unity, show that $xyz =a^3+b^3$

5) If w is a complex cube-root of unity, then prove the following

i) $(w^2+w-1)^3 = -8$

ii) $(a+b)+(aw+bw^2)+(aw^2+bw)= 0$

Let's remember!

* A number of the form a+ib, where a and b are real numbers, i = $\sqrt{-1}$, is called a complex number.

* Let $z_1 = a+ib$ and $z_2 = c+id$. Then

  $z_1 + z_2 = (a+c) + (b+d)i$

  $z_1 z_2 = (ac-bd) + (ad+bc)i$

* For any positive integer k,

  $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

* The conjugate of complex number $z = a+ib$ denoted by $\bar{z}$, is given by $\bar{z} = a-ib$

* The cube roots of unity are denoted by 1, w, w^2 or 1, w, $\bar{w}$

MISCELLANEOUS EXERCISE - 3

1) Find the value of is $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}$

2) Find the value of $\sqrt{-3} \times \sqrt{-6}$

3) Simplify the following and express in the form a+ib.

i) $3+\sqrt{-64}$ ii) $(2i^3)^2$ iii) $(2+3i)(1-4i)$

iv) $\dfrac{5}{2}i(-4-3i)$ v) $(1+3i)^2(3+i)$ vi) $\dfrac{4+3i}{1-i}$

vii) $\left(1+\dfrac{2}{i}\right)\left(3+\dfrac{4}{i}\right)(5+i)^{-1}$ viii) $\dfrac{\sqrt{5}+\sqrt{3}i}{\sqrt{5}-\sqrt{3}i}$

ix) $\dfrac{3i^5+2i^7+i^9}{i^6+2i^8+3i^{18}}$ x) $\dfrac{5+7i}{4+3i}+\dfrac{5+7i}{4-3i}$

4) Solve the following equations for $x, y \in R$

i) $(4-5i)x + (2+3i)y = 10-7i$

ii) $(1-3i)x + (2+5i)y = 7+i$

iii) $\dfrac{x+iy}{2+3i} = 7-i$

iv) $(x+iy)\ (5+6i) = 2+3i$

v) $2x+i^9y\ (2+i) = xi^7+10i^{16}$

5) Find the value of

i) $x^3+2x^2-3x+21$, if $x = 1+2i$.

ii) x^3-5x^2+4x+8, if $x = \dfrac{10}{3-i}$

iii) $x^3-3x^2+19x-20$, if $x = 1-4i$.

6) Find the square roots of

i) $-16+30i$ ii) $15-8i$ iii) $2+2\sqrt{3}\,i$

iv) $18i$ v) $3-4i$ vi) $6+8i$

ACTIVITY

Activity 3.1:

Carry out the following activity.

If $x + 2i = 7x-15i^6y + i^3(y + 4)$, find $x + y$.

Given : $x + 2i = 7x-15i^6y + i^3(y + 4)$

$x + 2i = 7x-15\,\square\,y + \square\,(y + 4)$

$x + 2i = 7x + \square\,y - (y + 4)\,\square$

$x - \square + 2i = 7x - (y + 4)\,\square$

$\therefore\ x - 15y - 7x = \square$ and $\square = -(y + 4)$

$\therefore\ -6x - 15y = \square$ and $y = \square$

$\therefore\ x = \square,\ y = \square$

$\therefore\ x + y = \square$

Activity 3.2:

Carry out the following activity

Find the value of $\dfrac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$ in terms of w^2

Consider $\dfrac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} = \dfrac{\square\left(\dfrac{a}{\omega^2}+\dfrac{b}{\omega}+c\right)}{c+a\omega+b\omega^2}$

$= \dfrac{\left(\dfrac{a\,\square}{\omega^3}+\dfrac{b\,\square}{\omega^3}+c\right)\omega^2}{c+a\omega+b\omega^2}$

$= \dfrac{\left(\dfrac{a\omega}{1}+\dfrac{b\,\square}{\square}+c\right)\omega^2}{c+a\omega+b\omega^2}$

$= \dfrac{(\square+\square+c)\,\omega^2}{c+a\omega+b\omega^2}$

$= \square$

❖ ❖ ❖