6 FUNCTIONS

Let's Study

- Function, Domain, Co-domain, Range
- Types of functions
- Representation of function
- Basic types of functions
- Piece-wise defined and special functions

Let's : Learn

6.1 Function

Definition : A function (or mapping) f from a set A to set B (f : A→B) is a relation which associates for each element x in A, a unique (exactly one) element y in B.

Then the element y is expressed as $y = f(x)$.

y is the image of x under f.

f is also called a map or transformation.

If such a function exists, then A is called the **domain** of f and B is called the **co-domain** of f.

Illustration:

Examine the following relations which are given by arrows of line segments joining elements in A and elements in B.

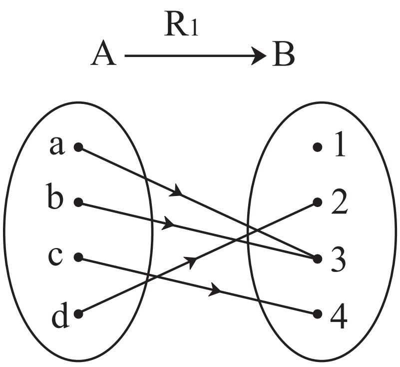


Fig. 6.1

Since, every element from A is associated to exactly one element in B, R_1 is a well defined function.

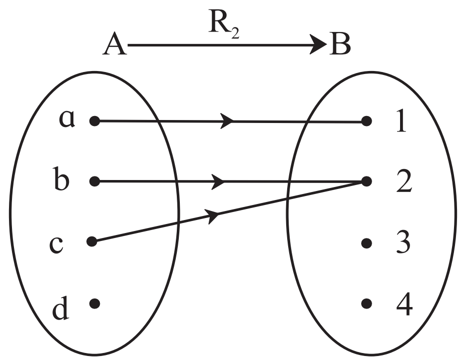


Fig. 6.2

R_2 is not a function because element 'd' in A is not associated to any element in B.

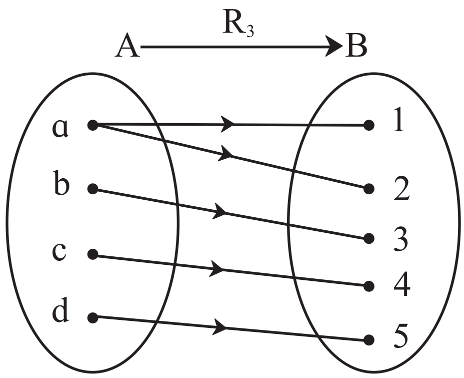


Fig. 6.3

R_3 is not a function because element a in A is associated to two elements in B.

The relation which defines a function f from domain A to co-domain B is often given by an algebraic rule.

For example, A = Z, the set of integers and B = Q the set of rational numbers and the function f is given by $f(n) = \frac{n}{7}$ here $n \in$ Z, f(n) $\in$ Q.

6.1.1 Types of function

One-one or **One to one** or **Injective function**

Definition : A function $f : A \to B$ is said to be one-one if different elements in A have different images in B. The condition is also expressed as

$f(a) = f(b) \Rightarrow a = b$ [As $a \neq b \Rightarrow f(a) \neq f(b)$]

Onto or **Surjective function**

Definition: A function $f : A \to B$ is said to be onto if every element y in B is an image of some x in A (or y in B has preimage x in A)

The image of A can be denoted by $f(A)$.

$f(A) = \{y \in B \mid y = f(x) \text{ for some } x \in A\}$

$f(A)$ is also called the **range** of f.

Note that $f : A \to B$ is onto if $f(A) = f$.

Also range of $f = f(A) \subset$ co-domain of f.

Illustration:

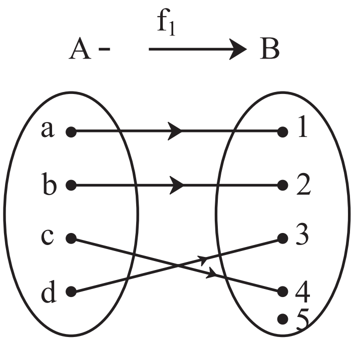


Fig. 6.4

f_1 is one-one, but not onto as element 5 is in B has no pre image in A

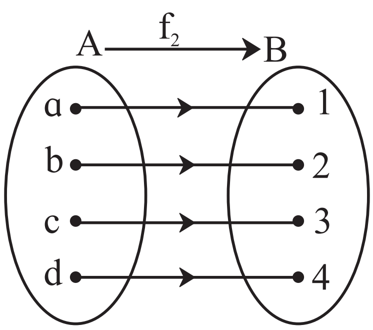


Fig. 6.5

f_2 is one-one, and onto

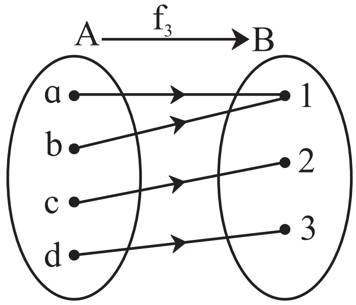


Fig. 6.6

f_3 is onto but not one-one as $f(a) = f(b) = 1$ but $a \neq b$.

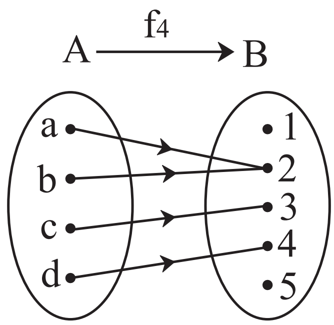
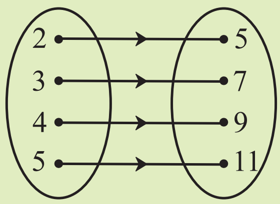


Fig. 6.7

f_4 is neither one-one, nor onto

6.1.2 Representation of Function

Verbal form	Output exceeds twice the input by 1 Domain : Set of inputs Range : Set of outputs
Arrow form on Venn Diagram	 **Fig. 6.8** Domain : Set of pre-images Range: Set of images
Ordered Pair (x, y)	$f = \{(2,5), (3,7), (4,9), (5,11)\}$ Domain : Set of 1st components from each ordered pair = $\{2, 3, 4, 5\}$ Range : Set of 2nd components from each ordered pair = $\{5, 7, 9, 11\}$

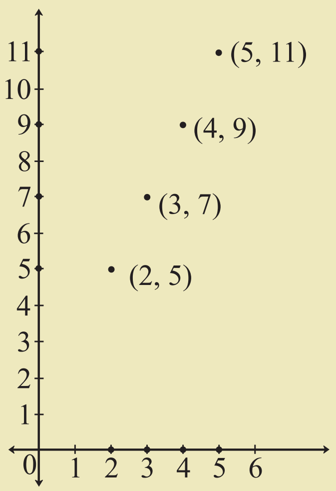
<table>
<tr><td>Rule / Formula</td><td>$y = f(x) = 2x + 1$
Where $x \in N$, $1 < x < 6$
$f(x)$ read as 'f of x' or 'function of x'
Domain : Set of values of x for which $f(x)$ is defined
Range : Set of values of y for which $f(x)$ is defined</td></tr>
<tr><td>Tabular Form</td><td><table><tr><td>x</td><td>y</td></tr><tr><td>2</td><td>5</td></tr><tr><td>3</td><td>7</td></tr><tr><td>4</td><td>9</td></tr><tr><td>5</td><td>11</td></tr></table>Domain : x values
Range: y values</td></tr>
<tr><td>Graphical form</td><td>

Fig. 6.9
Domain: Projection of graph on x-axis.
Range: Projection of graph on y-axis.</td></tr>
</table>

6.1.3 Graph of a function:

If the domain of function is in R, we can show the function by a graph in xy plane. The graph consists of points (x,y), where $y = f(x)$.

Vertical Line Test

Given a graph, let us find if the graph represents a function of x i.e. $f(x)$

A graph represents function of x, only if no vertical line intersects the curve in more than one point.

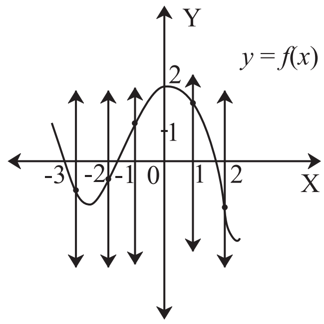


Fig. 6.10

Since every x has a unique associated value of y.

It is a function.

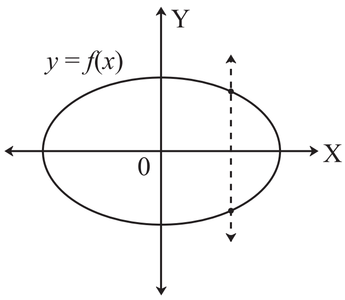


Fig. 6.11

This graph does not represent a function as vertical line intersects at more than one point some x has more than one values of y.

Horizontal Line Test:

If no horizontal line intersects the graph of a function in more than one point, then the function is one-one function.

Illustration:

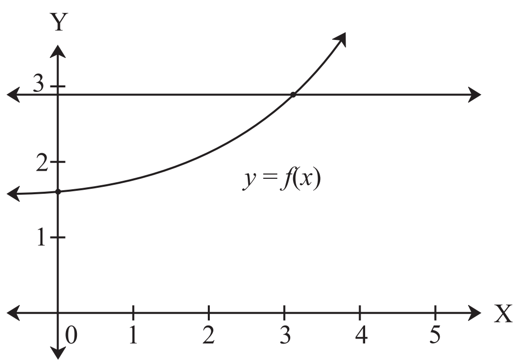


Fig. 6.12

The graph is a one-one function as a horizontal line intersects the graph at only one point.

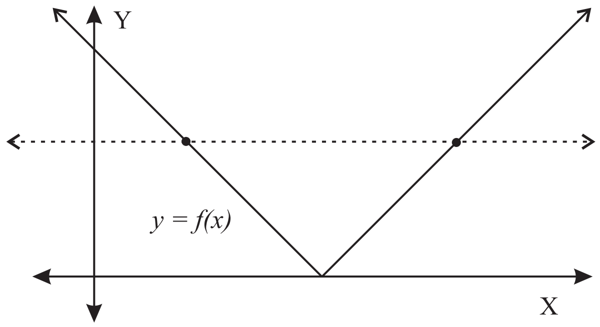


Fig. 6.13

The graph is a one-one function

6.1.4 Value of funcation : f(a) is called the value of funcation $f(x)$ at x = a

Evaluation of function:

Ex. 1) Evaluate $f(x) = 2x^2 - 3x + 4$ at

$$x = 7 \ \& \ x = -2t$$

Solution : $f(x)$ at $x = 7$ is $f(7)$

$$f(7) = 2(7)^2 - 3(7) + 4$$
$$= 2(49) - 21 + 4$$
$$= 98 - 21 + 4$$
$$= 81$$

$$f(-2t) = 2(-2t)^2 - 3(-2t) + 4$$
$$= 2(4t^2) + 6t + 4$$
$$= 8t^2 + 6t + 4$$

Ex. 2) Using the graph of $y = g(x)$, find $g(-4)$ and $g(3)$

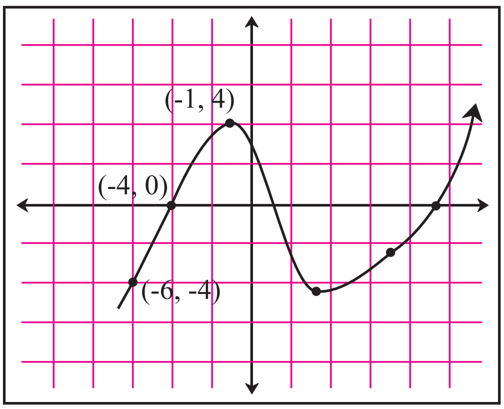


Fig. 6.14

Solution : From graph when $x = -4$, $y = 0$

so $g(-4) = 0$

From graph when $x = 3$, $y = -5$ so $g(3) = -5$

Function Solution:

Ex. 3) If $t(m) = 3m^2 - m$ and $t(m) = 4$, then find m

Solution : As

$$t(m) = 4$$
$$3m^2 - m = 4$$
$$3m^2 - m - 4 = 0$$
$$3m^2 - 4m + 3m - 4 = 0$$
$$m(3m - 4) + 1(3m - 4) = 0$$
$$(3m - 4)(m + 1) = 0$$

Therefore , $m = \dfrac{4}{3}$ or $m = -1$

Ex. 4) From the graph below find x for which $f(x) = 4$

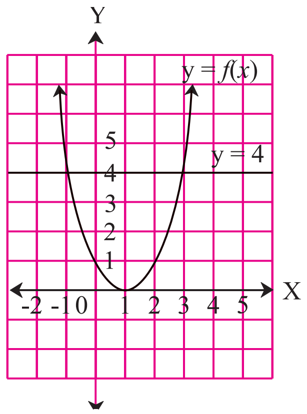


Fig. 6.15

Solution : To solve $f(x) = 4$ i.e. $y = 4$

Find the values of x where graph intersects line $y = 4$

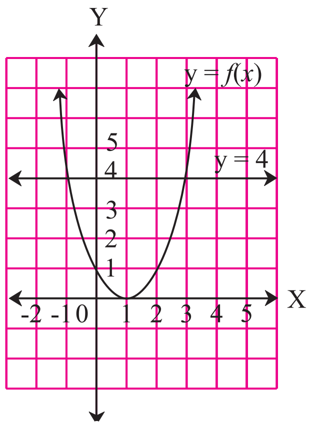


Fig. 6.16

Therefore , $x = -1$ and $x = 3$.

Function from equation:

Ex. 5) (Activity) From the equation $4x + 7y = 1$ express

i) y as a function of x

ii) x as a function of y

Solution : Given equation is $4x + 7y = 1$

i) From the given equation

$7y = \square$

$y = \square$ = function of x

So $y = f(x) = \square$

ii) From the given equation

$4x = \square$

$x = \square$ = function of y

So $x = g(y) = \square$

6.1.5 Some Basic Functions

(Here f : R $\rightarrow$ R Unless stated otherwise)

1) Constant Function

Form : $f(x) = k$, $k \in R$

Example : Graph of $f(x) = 2$

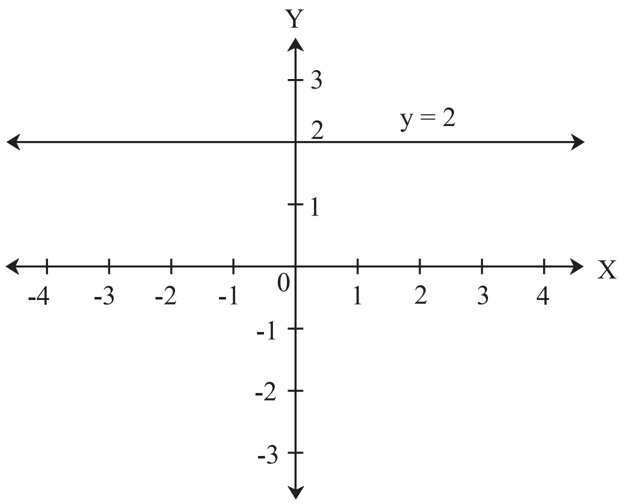


Fig. 6.17

Domain : R or $(-\infty, \infty)$ and **Range :** $\{2\}$

2) Identity function

If f : R $\rightarrow$ R then identity function is defined by $f(x) = x$, for every $x \in R$.

Identity function is given in the graph below.

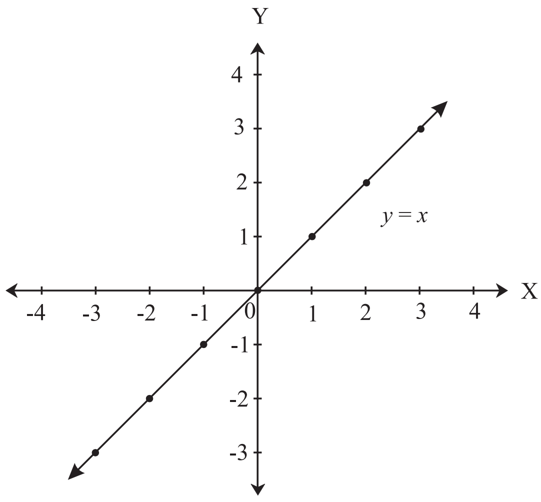


Fig. 6.18

Domain : R or $(-\infty, \infty)$ and **Range :** R or $(-\infty, \infty)$

[**Note :** Identity function is also given by I $(x) = x$].

3) Power Functions : $f(x) = ax^n$, $n \in N$

(Note that this function is a multiple of n^th power of x)

i) Square Function

Example : $f(x) = x^2$

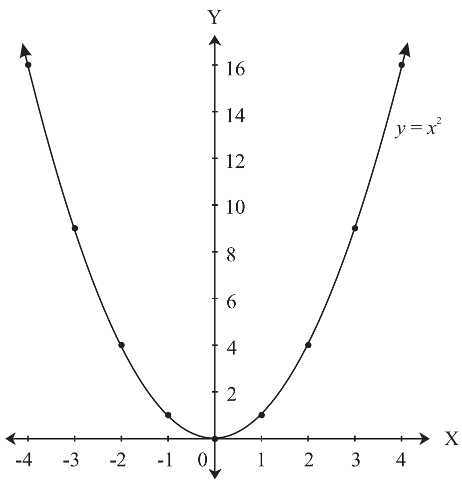


Fig. 6.19

Domain : R or $(-\infty, \infty)$ and **Range :** $[0, \infty)$

Properties:

1) Graph of $f(x) = x^2$ is a parabola opening upwards and with vertex at origin.

2) Graph is symmetric about y - axis .

3) The graph of even powers of x looks similar to square function. (verify !) e.g. x^4, x^6.

4) $(y - k) = (x - h)^2$ represents parabola with vertex at (h, k)

5) If $-2 \le x \le 2$ then $0 \le x^2 \le 4$ (see fig.) and if $-3 \le x \le 2$ then $0 \le x^2 \le 9$ (see fig).

ii) Cube Function

Example : $f(x) = x^3$

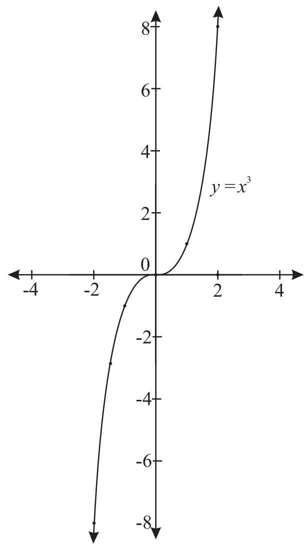


Fig. 6.20

Domain : R or $(-\infty, \infty)$ and **Range :** R or $(-\infty, \infty)$

Properties:

1) The graph of odd powers of x (more than 1) looks similar to cube function. e.g. x^5, x^7.

4) Polynomial Function

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$$

is polynomial function of degree n , if $a_0 \ne 0$, and a_i s are real.

i) Linear Function

Form : $f(x) = ax + b \ (a \ne 0)$

Example : $f(x) = -2x + 3, \ x \in R$

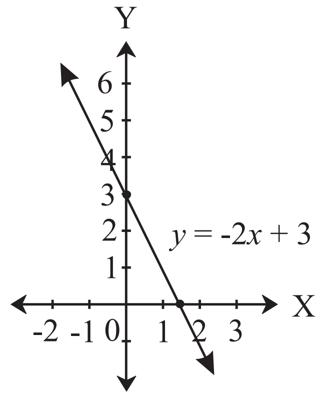


Fig. 6.21

Domain : R or $(-\infty, \infty)$ and **Range :** R or $(-\infty, \infty)$

Properties :

1) Graph of $f(x) = ax + b$ is a line with slope 'a', y-intercept 'b' and x-intercept $\left(-\dfrac{b}{a}\right)$.

2) Function : is increasing when slope is positive and deceasing when slope is negative.

ii) Quadratic Function

Form : $f(x) = ax^2 + bx + c \ (a \ne 0)$

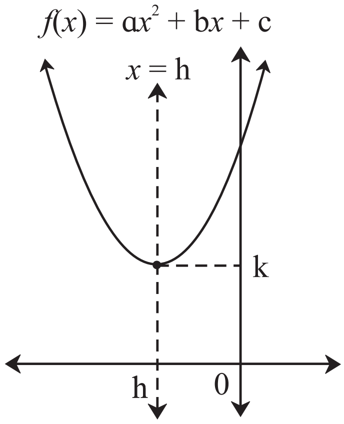


Fig. 6.22

Domain : R or $(-\infty, \infty)$ and **Range :** $[k, \infty)$

Properties :

1) Graph of $f(x) = ax^2 + bx + c$ and where $a > 0$ is a parabola.

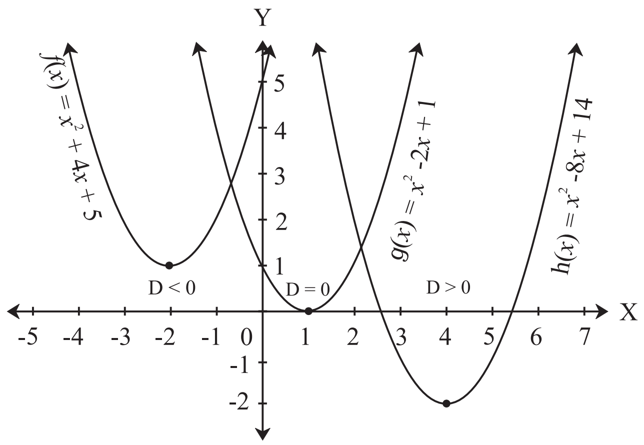


Fig. 6.23

Consider, $y = ax^2 + bx + c$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a}$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a}$$

$$\left(y + \frac{b^2-4ac}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$

With change of variable

$$X = x + \frac{b}{2a}, \ Y = y + \frac{b^2-4ac}{4a}$$

this is a parabola Y = aX^2

This is a parabola with vertex

$\left(-\frac{b}{2a}, \frac{b^2-4ac}{4a}\right)$ or $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ where

D = b^2 – 4ac and the parabola is opening upwards.

There are three possibilities.

For a>0,

i) If D = b^2 – 4ac = 0, the parabola touches x-axis and $y \geq 0$ for all x. e.g. $g(x) = x^2 - 2x + 1$

ii) If D = b^2 – 4ac > 0, then parabola intersects x-axis at 2 distinct points. Here y is negative for values of x between the 2 roots and positive for large or small x.

iii) If D = b^2 – 4ac < 0, the parabola lies above x-axis and $y \neq 0$ for any x. Here y is positive for all values of x. e.g. $f(x) = x^2 + 4x + 5$

iii) Cubic Function

Example : $f(x) = ax^3 + bx^2 + cx + d \ (a \neq 0)$

Domain : R or $(-\infty, \infty)$ and

Range : R or $(-\infty, \infty)$

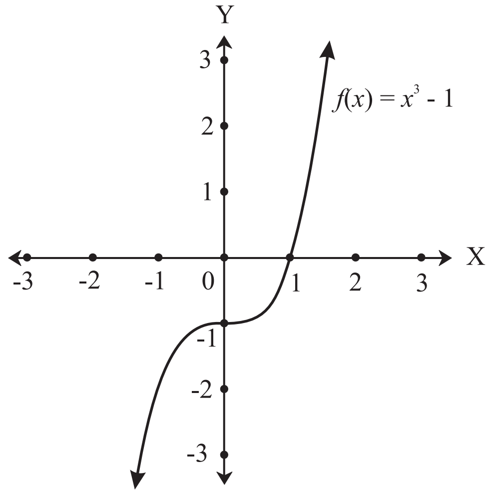


Fig. 6.24

Property:

1) Graph of $f(x) = x^3 - 1$

$f(x) = (x - 1)(x^2 + x + 1)$ cuts x-axis at only one point (1,0) , which means $f(x)$ has one real root & two complex roots.

Note that, any polynomial of odd degree must have at least one real root, because the complex roots appear in conjugate pairs.

5) Radical Function

Ex: $f(x) = \sqrt[n]{x}$, n ∈ N

1. Square root function

$f(x) = \sqrt{x}, x \geq 0$

(Since square root of negative number is not a real number, so the domain of $\sqrt{x}$ is restricted to positive values of x).

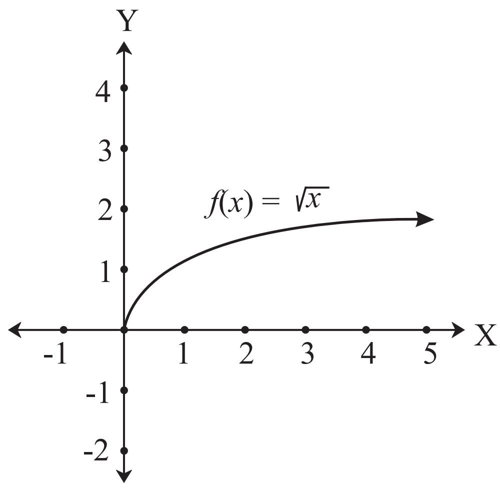


Fig. 6.25

Domain : $[0, \infty)$ and **Range :** $[0, \infty)$

Note :

1) If x is positive, there are two square roots of x. By convention $\sqrt{x}$ is positive root and $-\sqrt{x}$ is the negattive root.

2) If $-4 < x < 9$, as $\sqrt{x}$ is only deifned for $x \geq 0$, so $0 \leq \sqrt{x} < 3$.

Ex. 6 : Find the domain and range of $f(x) = \sqrt{9-x^2}$.

Soln. : $f(x) = \sqrt{9-x^2}$ is defined for

$9 - x^2 \geq 0$, i.e. $x^2 - 9 \leq 0$ i.e. $(x-3)(x+3) \leq 0$

Therefore $[-3, 3]$ is domain of $f(x)$. (Verify !)

To find range, let $\sqrt{9-x^2} = y$

Since square root is always positive, so $y \geq 0$ ...(I)

Also, on squaring we get $9 - x^2 = y^2$

Since, $3 \leq x \leq 3$

i.e. $0 \leq x^2 \leq 9$

i.e. $0 \geq -x^2 \geq -9$

i.e. $9 \geq 9 - x^2 \geq 9 - 9$

i.e. $9 \geq 9 - x^2 \geq 0$

i.e. $3 \geq \sqrt{9-x^2} \geq 0$

$\therefore\ 3 \geq y \geq 0$ ...(II)

From (I) and (II), $y \in [0,3]$ is range of $f(x)$.

2. Cube root function

$f(x) = \sqrt[3]{x}$,

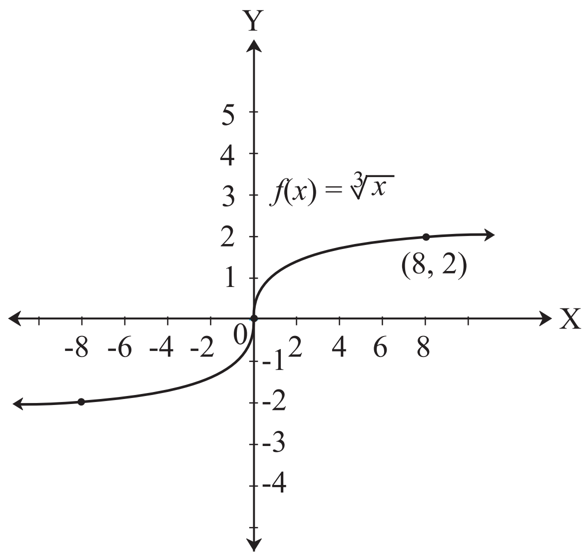


Fig. 6.26

Domain : R and **Range :** R

Note : If $-8 \leq x \leq 1$ then $-2 \leq \sqrt[3]{x} \leq 1$.

Ex. 7 : Find the domain $f(x) = \sqrt{x^3 - 8}$.

Soln. : $f(x)$ is defined for $x^3 - 8 \geq 0$

i.e. $x^3 - 2^3 \geq 0$, $(x-2)(x^2 + 2x + 4) \geq 0$

In $x^2 + 2x + 4$, $a = 1 > 0$ and $D = b^2 - 4ac = 2^2 - 4 \times 1 \times 4 = -12 < 0$

Therefore, $x^2 + 2x + 4$ is a positive quadratic.

i. e. $x^2 + 2x + 4 > 0$ for all x

Therefore $x - 2 \geq 0$, $x \geq 2$ is the domain.

i.e. Domain is $x \in [2, \infty)$

6) Rational Function

Definition: Given polynomials

$p(x), q(x)\ f(x) = \dfrac{p(x)}{q(x)}$ is defined for x if $q(x) \neq 0$.

Example : $f(x) = \dfrac{1}{x}$, $x \neq 0$

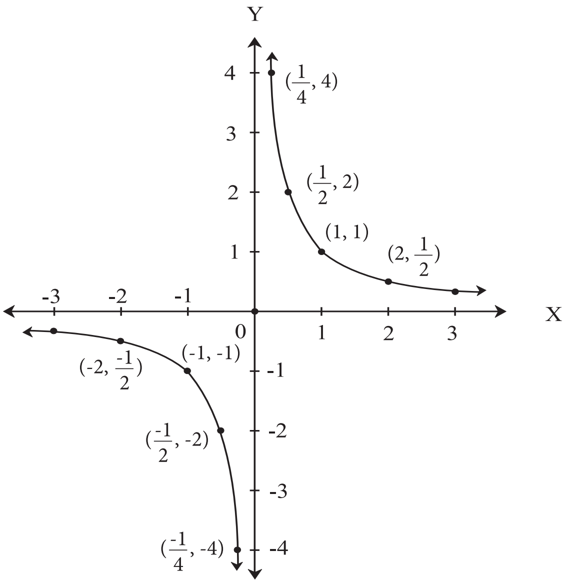


Fig. 6.27

Domain : R-{0} and **Range :** $R - \{0\}$

Properties:

1) As $x \to 0$ i.e. (As x approaches 0) $f(x) \to \infty$ or $f(x) \to -\infty$, so the line $x = 0$ i.e y-axis is called vertical asymptote.(A straight line which does not intersect the curve but as x approaches to ∞ or $-\infty$ the distance between the line and the curve tends to 0, is called an asymptote of the curve.)

2) As As $x \to \infty$ or $x \to -\infty$, $f(x) \to 0$, $y = 0$ the line i.e y-axis is called horizontal asymptote.

3) The domain of rational function $f(x) = \dfrac{p(x)}{q(x)}$ is all the real values of except the zeroes of $q(x)$.

Ex. 8 : Find domain and range of the function $f(x) = \dfrac{6-4x^2}{4x+5}$

Solution : $f(x)$ is defined for all $x \in R$ exccept when denominators is 0.

Since, $4x + 5 = 0 \Rightarrow x = -\dfrac{5}{4}$.

So Domain of $f(x)$ is $R - \left\{-\dfrac{5}{4}\right\}$.

To find the range, let $y = \dfrac{6-4x^2}{4x+5}$

i.e. $y(4x + 5) = 6 - 4x^2$

i.e. $4x^2 + (4y)x + 5y - 6 = 0$.

This is a quadratic equation in x with y as constant.

Since $x \in R - \{-5/4\}$, i.e. x is real, we get Solution if, $D = b^2 - 4ac \geq 0$

i.e. $(4y)^2 - 4(4)(5y - 6) \geq 0$

$16y^2 - 16(5y - 6) \geq 0$

$y^2 - 5y + 6 \geq 0$

$(y - 2)(y - 3) \geq 0$

Therefore $y \leq 2$ or $y \geq 3$ (Verify!)

Range of $f(x)$ is $(-\infty, 2] \cup [3,\infty)$

7) Exponential Function

Form : $f(x) = a^x$ is an exponential function with base a and exponent (or index) x, $a \neq 0$, $a > 0$ and $x \in R$.

Example : $f(x) = 2^x$ and $f(x) = 2^{-x}$

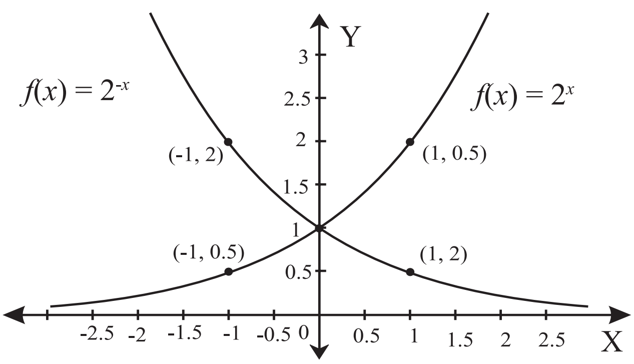


Fig. 6.28

Domain: R and **Range :** $(0, \infty)$

Properties:

1) As $x \to -\infty$, then $f(x) = 2^x \to 0$, so the graph has horizontal asymptote $(y = 0)$

2) By taking the natural base e ($\approx$ 2.718), graph of $f(x) = e^x$ is similar to that of 2^x in appearance

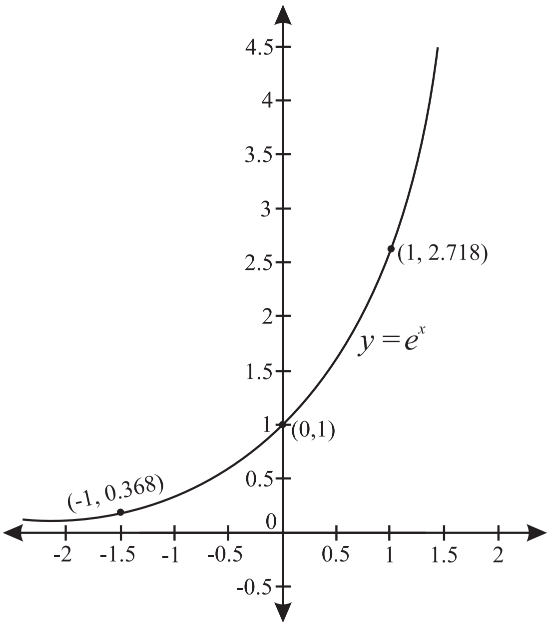


Fig. 6.29

3) For $a > 0$, $a \neq 1$, if $a^x = a^y$ then $x = y$. So a^x is one-one function. (check graph for horizontal line test).

4) $r > 1, m > n \Rightarrow r^m > r^n$ and
$r < 1, m > n \Rightarrow r^m < r^n$

Ex. 9 : Solve $5^{2x+7} = 125$.

Solution : As $5^{2x+7} = 125$

i.e : $5^{2x+7} = 5^3$, $\therefore\ 2x + 7 = 3$

and $x = \dfrac{3-7}{2} = \dfrac{-4}{2} = -2$

Ex. 10 : Find the domain of $f(x) = \sqrt{6 - 2^x - 2^{3-x}}$

Solution : Since $\sqrt{x}$ is defined for $x \geq 0$

$f(x)$ is defined for $6 - 2^x - 2^{3-x} \geq 0$

i.e. $6 - 2^x - \dfrac{2^3}{2^x} \geq 0$

i.e. $6.2^x - (2^x)^2 - 8 \geq 0$

i.e. $(2^x)^2 - 6.2^x + 8 \leq 0$

i.e. $(2^x - 4)(2^x - 2) \leq 0$

$2^x \geq 2$ and $2^x \leq 4$ (Verify !)

$2^x \geq 2^1$ and $2^x \leq 2^2$

$x \geq 1$ and $x \leq 2$ or $1 \leq x \leq 2$

Doamin is [1,2]

8) Logarithmic Function:

Let, $a > 0$, $a \neq 1$, Then logarithmic function $\log_a x$, $y = \log_a x$ if $x = a^y$.

for $x > 0$, is defined as

$$\underset{\text{log arithmic form}}{y = \log_a x} \Leftrightarrow \underset{\text{exponential form}}{a^y = x}$$

Properties:

1) As $a^0 = 1$, so $\log_a 1 = 0$ and as $a^1 = a$,

so $\log_a a = 1$

2) As $a^x = a^y \Leftrightarrow x = y$ so $\log_a x = \log_a y \Leftrightarrow x = y$

3) Product rule of logarithms.

For $a, b, c > 0$ and $a \neq 1$,

$\log_a bc = \log_a b + \log_a c$ (Verify !)

4) Quotient rule of logarithms.

For $a, b, c > 0$ and $a \neq 1$,

$\log_a \dfrac{b}{c} = \log_a b - \log_a c$ (Verify !)

5) Power/Exponent rule of logarithms.

For $a, b, c > 0$ and $a \neq 1$,

$\log_a b^c = c \log_a b$ (Verify !)

6) For natural base e , $\log_e x = ln\ x$ as Natural Logarithm Function.

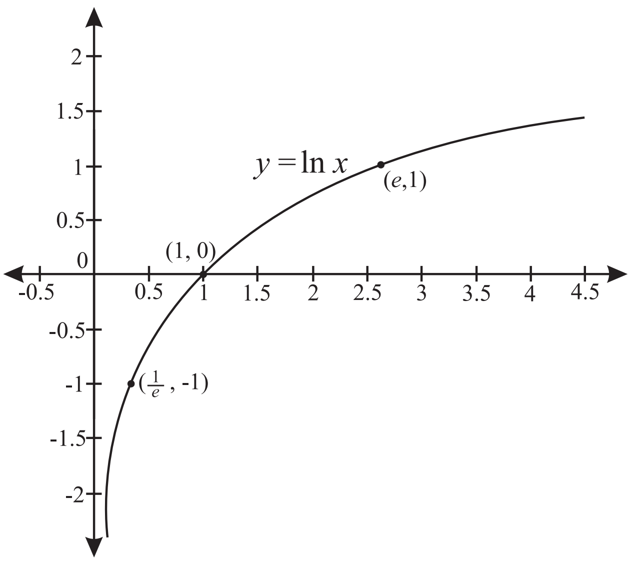


Fig. 6.30

Here domain of $ln\ x$ is $(0, \infty)$ and range is $(-\infty, \infty)$.

8) Logarithmic inequalities:

(i) If $a>1$, $0<m<n$ then $\log_a m < \log_a n$

e.g. $\log_{10} 20 < \log_{10} 30$

(ii) If $0>a<1$, $0<m<n$ then $\log_a m > \log_a n$

e.g. $\log_{0.1} 20 > \log_{0.1} 30$

(iii) For a, $m>0$ if a and m lies on the same side of unity (i.e. 1) then $\log_a m>0$.

e.g. $\log_2 3>0$, $\log_{0.3} 0.5>0$

(iv) For a, $m>0$ if a and m lies on the different sides of unity (i.e. 1) then $\log_a m>0$.

e.g. $\log_{0.2} 3<0$, $\log_3 0.5>0$

Ex. 11 : Write log72 in terms of log2 and log3.

Solution : $\log 72 = \log(2^3.3^2)$

$= \log 2^3 + \log 3^2$ ($\because$ Power rule)

$= 3 \log 2 + 2 \log 3$($\because$ Power rule)

Ex. 12 : Evaluate $\ln e^9 - \ln e^4$.

Solution : $\ln e^9 - \ln e^4 = \log_e e^9 - \log_e e^4$

$= 9 \log_e e - 4 \log_e e$

$= 9(1) - 4(1)$ $(\therefore \ln e = 1)$

$= 5$

Ex. 13 : Expand $\log\left[\dfrac{x^3(x+3)}{2(x-4)^2}\right]$

Solution : Using Quotient rule

$= \log [x^3(x+3)] - \log [2 (x-4)^2]$

Using Product rule

$= [\log x^3 + \log (x+3)] - [\log 2 + \log (x-4)^2]$

Using Power rule

$= [3\log x + \log (x+3)] - [\log 2 - 2\log (x-4)]$

$= 3\log x + \log (x+3) - \log 2 + 2\log (x-4)$

Ex. 14 : Combine

$3\ln (p + 1) - \frac{1}{2} \ln r + 5\ln(2q + 3)$ into single logarithm.

Solution : Using Power rule,

$= \ln (p + 1)^3 - \ln r^{\frac{1}{2}} + \ln(2q +3)^5$

Using Quotient rule

$= \ln \dfrac{(p+1)^3}{\sqrt{r}} + \ln(2q +3)^5$

Using Product rule

$= \ln \left[\dfrac{(p+1)^3}{\sqrt{r}}(2q+3)^5\right]$

Ex. 15 : Find the domain of $ln(x - 5)$.

Solution : As $ln(x - 5)$ is defined for $(x - 5) > 0$

that is $x > 5$ so domain is $(5, \infty)$.

Let's note:

1) $\log(x + y) \neq \log x + \log y$

2) $\log x \log y \neq \log(xy)$

3) $\dfrac{\log\ x}{\log\ y} \neq \log\left(\dfrac{x}{y}\right)$

4) $(\log x)^n \neq n \log^n$

9) Change of base formula:

For $a, x, b > 0$ and $a,\ b \neq 1$, $\log_a x = \dfrac{\log_b\ x}{\log_b\ a}$

Note: $\log_a x = \dfrac{\log_x\ x}{\log_x\ a} = \dfrac{1}{\log_x\ a}$ (Verify !)

Ex. 16 : Evaluate $\dfrac{\log_4\ 81}{\log_4\ 9}$

Solution : By Change of base law, as the base is same (that is 4)

$$\frac{\log_4\ 81}{\log_4\ 9} = \log_9 81 = 2$$

Ex. 17 : Prove that, $2\log_b a^4 .\ \log_c b^3 .\ \log_a c^5 = 120$

Solution : L.H.S. $= 2\log_b a^4 .\ \log_c b^3 .\ \log_a c^5$

$= 4 \times 2\log_b a \times 3\log_c b \times 5\log_a c$

Using change of base law,

$= 4 \times 2\ \dfrac{\log a}{\log b} \times 3\ \dfrac{\log b}{\log c} \times 5\ \dfrac{\log c}{\log a}$

$= 120$

Ex. 18 : Find the domain of $f(x) = \log_{x+5}(x^2 - 4)$

Solution : SInce $\log_a x$ is definedd for a, $x > 0$ and a $\neq 1$ $f(x)$ is defined for $(x^2 - 4) > 0$, $x + 5 > 0$, $x + 5 \neq 1$.

i.e. $(x - 2)(x + 2) > 0$, $x > -5$, $x \neq -4$

i.e. $x < -2$ or $x > 2$ and $x > -5$ and $x \neq -4$

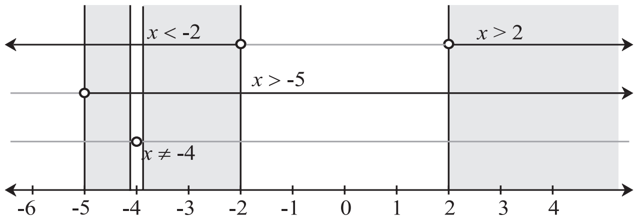


Fig. 6.31

9) Trigonometric function

The graphs of trigonometric functions are discusse in chapter 2 of Mathematics Book I.

$f(x)$	Domain	Range
$\sin x$	R	$[-1,1]$
$\cos x$	R	$[-1,1]$
$\tan x$	$R - \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}...\right\}$	R

EXERCISE 6.1

1) Check if the following relations are functions.

(a)

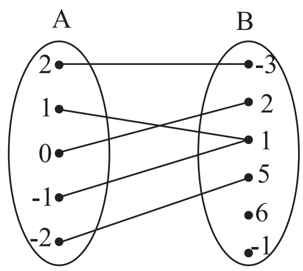


Fig. 6.32

(b)

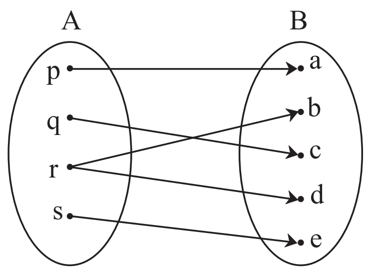


Fig. 6.33

(c)

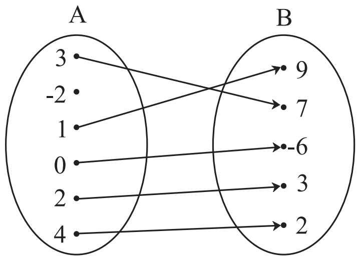


Fig. 6.34

2) Which sets of ordered pairs represent functions from A = {1, 2, 3, 4} to B = {−1, 0, 1, 2, 3}? Justify.

(a) {(1,0), (3,3), (2,−1), (4,1), (2,2)}

(b) {(1,2), (2,−1), (3,1), (4,3)}

(c) {(1,3), (4,1), (2,2)}

(d) {(1,1), (2,1), (3,1), (4,1)}

3) Check if the relation given by the equation represents y as function of x.

(a) $2x + 3y = 12$ (b) $x + y^2 = 9$

(c) $x^2 - y = 25$ (d) $2y + 10 = 0$

(e) $3x - 6 = 21$

4) If $f(m) = m^2 - 3m + 1$, find

(a) $f(0)$ (b) $f(-3)$

(c) $f\left(\frac{1}{2}\right)$ (d) $f(x + 1)$

(e) $f(-x)$

(f) $\left(\frac{f(2+h)-f(2)}{h}\right), h \neq 0$.

5) Find x, if $g(x) = 0$ where

(a) $g(x) = \frac{5x-6}{7}$ (b) $g(x) = \frac{18-2x^2}{7}$

(c) $g(x) = 6x^2 + x - 2$

(d) $g(x) = x^3 - 2x^2 - 5x + 6$

6) Find x, if $f(x) = g(x)$ where

(a) $f(x) = x^4 + 2x^2$, $g(x) = 11x^2$

(b) $f(x) = \sqrt{x} - 3$, $g(x) = 5 - x$

7) If $f(x) = \frac{a-x}{b-x}$, $f(2)$ is undefined, and $f(3) = 5$, find a and b.

8) Find the domain and range of the follwoing functions.

(a) $f(x) = 7x^2 + 4x - 1$

(b) $g(x) = \frac{x+4}{x-2}$

(c) $h(x) = \frac{\sqrt{x+5}}{5+x}$

(d) $f(x) = \sqrt[3]{x+1}$

(e) $f(x) = \sqrt{(x-2)(5-x)}$

(f) $f(x) = \sqrt{\frac{x-3}{7-x}}$

(g) $f(x) = \sqrt{16-x^2}$

9) Express the area A of a square as a function of its (a) side s (b) perimeter P.

10) Express the area A of circle as a function of its (a) radius r (b) diameter d (c) circumference C.

11) An open box is made from a square of cardboard of 30 cms side, by cutting squares of length x centimeters from each corner and folding the sides up. Express the volume of the box as a function of x. Also find its domain.

Let f be a subset of Z × Z defined by

12) $f = \{(ab, a+b) : a, b \in Z\}$. Is f a function from Z to Z? Justify.

14) Check the injectivity and surjectivity of the following functions.

(a) $f : N \to N$ given by $f(x) = x^2$

(b) $f : Z \to Z$ given by $f(x) = x^2$

(c) $f : R \to R$ given by $f(x) = x^2$

(d) $f : \mathrm{N} \to \mathrm{N}$ given by $f(x) = x^3$

(e) $f : \mathrm{R} \to \mathrm{R}$ given by $f(x) = x^3$

14) Show that if $f : A \to B$ and $g : B \to C$ are one-one, then $g \circ f$ is also one-one.

15) Show that if $f : A \to B$ and $g : B \to C$ are onto, then $g \circ f$ is also onto.

16) If $f(x) = 3(4^{x+1})$ find $f(-3)$.

17) Express the following exponential equations in logarithmic form

(a)$2^5 = 32$ (b) $54^0 = 1$

(c) $23^1 = 23$ (d) $9^{3/2} = 27$

(e) $3^{-4} = \dfrac{1}{81}$ (f) $10^{-2} = 0.01$

(g) $e^2 = 7.3890$ (h) $e^{1/2} = 1.6487$

(i) $e^{-x} = 6$

18) Express the following logarithmic equations in exponential form

(a) $\log_2 64 = 6$ (b) $\log_5 \dfrac{1}{25} = -2$

(c) $\log_{10} 0.001 = -3$ (d) $\log_{1/2}(-8) = 3$

(e) $\ln 1 = 0$ (f) $\ln e = 1$

(g) $\ln \dfrac{1}{2} = -0.693$

19) Find the domain of

(a) $f(x) = \ln(x-5)$

(b) $f(x) = \log_{10}(x^2 - 5x + 6)$

20) Write the following expressions as sum or difference of logarithms

(a) $\log\left(\dfrac{pq}{rs}\right)$ (b) $\log\left(\sqrt{x}\sqrt[3]{y}\right)$

(c) $\ln\left(\dfrac{a^3(a-2)^2}{\sqrt{b^2+5}}\right)$

(d) $\ln\left[\dfrac{\sqrt[3]{x-2}\,(2x+1)^4}{(x+4)\sqrt{2x+4}}\right]^2$

21) Write the following expressions as a single logarithm.

(a) $5\log x + 7\log y - \log z$

(b) $\dfrac{1}{3}\log(x-1) + \dfrac{1}{2}\log(x)$

(c) $\ln(x+2) + \ln(x-2) - 3\ln(x+5)$

22) Given that $\log 2 = a$ and $\log 3 = b$, write $\log\sqrt{96}$ in terms of a and b.

23) Prove that

(a) $b^{\log_b a} = a$ (b) $\log_{b^m} a = \dfrac{1}{m}\log_b a$

(c) $a^{\log_c b} = b^{\log_c a}$

24) If $f(x) = ax^2 - bx + 6$ and $f(2) = 3$ and $f(4) = 30$, find a and b

25) Solve for x.

(a) $\log 2 + \log(x+3) - \log(3x-5) = \log 3$

(b) $2\log_{10} x = 1 + \log_{10}\left(x + \dfrac{11}{10}\right)$

(c) $\log_2 x + \log_4 x + \log_{16} x = \dfrac{21}{4}$

(d) $x + \log_{10}(1+2^x) = x\log_{10}5 + \log_{10}6$

26) If $\log\left(\dfrac{x+y}{3}\right) = \dfrac{1}{2}\log x + \dfrac{1}{2}\log y$, show that $\dfrac{x}{y} + \dfrac{y}{x} = 7$.

27) If $\log\left(\dfrac{x-y}{4}\right) = \log\sqrt{x} + \log\sqrt{y}$, show that $(x+y)^2 = 20xy$

28) If $x = \log_a bc$, $y = \log_b ca$, $z = \log_c ab$ then prove that $\dfrac{1}{1+x} + \dfrac{1}{1+y} + \dfrac{1}{1+z} = 1$

6.2 Algebra of functions:

Let f and g be functions with domains A and B. Then the functions $f+g, f-g, fg, \frac{f}{g}$ are defined on $A \cap B$ as follows.

Operations
$(f+g)(x) = f(x) + g(x)$
$(f-g)(x) = f(x) - g(x)$
$(f.g)(x) = f(x).g(x)$
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ where $g(x) \neq 0$

Ex. 1 : If $f(x) = x^2 + 2$ and $g(x) = 5x - 8$, then find

i) $(f+g)(1)$

ii) $(f-g)(-2)$

ii) $(f \circ g)(3m)$

iv) $\frac{f}{g}(0)$

Solution : i) As $(f+g)(x) = f(x) + g(x)$

$(f+g)(1) = f(1) + g(1)$

$= [(1)^2 + 2] + [5(1) - 8]$

$= 3 + (-3)$

$= 0$

ii) As $(f-g)(x) = f(x) - g(x)$

$(f-g)(-2) = f(-2) - g(-2)$

$= [(-2)^2 + 2] - [5(-2) - 8]$

$= [4 + 2] - [-10-8]$

$= 6 + 18$

$= 24$

iii) As $(fg)(x) = f(x)\,g(x)$

$(f \circ g)(3m) = f(3m)g(3m)$

$= [(3m)^2 + 2]\,[5(3m) - 8]$

$= [9m^2 + 2]\,[15m-8]$

$= 135m^3 - 72m^2 + 30m - 16$

iv) As $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$

$$\left(\frac{f}{g}\right)(0) = \frac{f(0)}{g(0)} = \frac{0^2+2}{5(0)-8}$$

$$= \frac{2}{-8} = -\frac{1}{4}$$

Ex. 2 : Given the function $f(x) = 5x^2$ and $g(x) = \sqrt{4-x}$ find the domain of

i) $(f+g)(x)$ ii) $(f \circ g)(x)$ iii) $\frac{f}{g}(x)$

Solution : i) Domain of $f(x) = 5x^2$ is $(-\infty, \infty)$.

To find domain of $g(x) = \sqrt{4-x}$

$4 - x \geq 0$

$x - 4 \leq 0$

Let $x \leq 4$, So domain is $(-\infty, 4]$.

Therefore, domain of $(f+g)(x)$ is

$(-\infty, \infty) \cap (-\infty, 4]$, that is $(-\infty, 4]$

ii) Similarly, domain of $(f \circ g)(x) = 5x^2\sqrt{4-x}$ is $(-\infty, 4]$

iii) And domain of $\left(\frac{f}{g}\right)(x) = \frac{5x^2}{\sqrt{4-x}}$ is $(-\infty, 4)$

As , at $x = 4$ the denominator $g(x) = 0$.

6.2.1 Composition of Functions:

A method of combining the function $f: A \to B$ with $g: B \to C$ is composition of functions, defined as $(f \circ g)(x) = f[g(x)]$ an read as 'f composed with g'

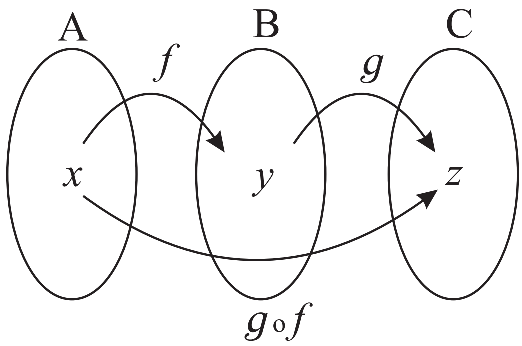


Fig. 6.35

Note:

1) The domain of $g \circ f$ is the set of all x in A such that $f(x)$ is in the B. The range of $g \circ f$ is set of all g $[f(x)]$ in C such that f(x) is in B.

2) Domain of $g \circ f \subseteq$ Domain of f and Range of $g \circ f \subseteq$ Range of g.

Illustration:

A cow produces 4 liters of milk in a day. Then x number of cows produce $4x$ liters of milk in a day. This is given by function $f(x) = 4x = 'y'$. Price of one liter milk is Rs. 50. Then the price of y liters of the milk is Rs. $50y$. This is given by another function $g(y) = 50y$. Now a function $h(x)$ gives the money earned from x number of cows in a day as a composite function of f and g as $h(x) = (g \circ f)(x) = g[f(x)] = g(4x) = 50(4x) = 200x$.

Ex. 3 : If $f(x) = \dfrac{2}{x+5}$ and $g(x) = x^2 - 1$, then find

i) $(f \circ g)(x)$ ii) $(g \circ f)(3)$

Solution :

i) As $(f \circ g)(x) = f[g(x)]$ and $f(x) = \dfrac{2}{x+5}$

Replace x from $f(x)$ by $g(x)$, to get

$$(f \circ g)(x) = \frac{2}{g(x)+5}$$

$$= \frac{2}{x^2-1+5}$$

$$= \frac{2}{x^2+4}$$

ii) As $(g \circ f)(x) = g[f(x)]$ and $g(x) = x^2 - 1$

Replace x by $f(x)$, to get

$$(g \circ f)(x) = [f(x)]^2 - 1$$

$$= \left(\frac{2}{x+5}\right)^2 - 1$$

Now let $x = 3$

$$(g \circ f)(3) = \left(\frac{2}{3+5}\right)^2 - 1$$

$$= \left(\frac{2}{8}\right)^2 - 1$$

$$= \left(\frac{1}{4}\right)^2 - 1$$

$$= \frac{1-16}{16}$$

$$= -\frac{15}{16}$$

Ex 4 : If $f(x) = x^2$, $g(x) = x + 5$, and $h(x) = \dfrac{1}{x}$, $x \neq 0$, find $(g \circ f \circ h)(x)$

Solution : $(g \circ f \circ h)(x)$

$$= g\{f[h(x)]$$

$$= g\left[f\left(\frac{1}{x}\right)\right]$$

$$= g\left[f\left(\frac{1}{x}\right)^2\right]$$

$$= \left(\frac{1}{x}\right)^2 + 5$$

$$= \frac{1}{x^2} + 5$$

Ex. 5 : If $h(x) = (x - 5)^2$, find the functions f and g, such that $h = f \circ g$.

→ In $h(x)$, 5 is subtracted from x first and then squared. Let $g(x) = x - 5$ and $f(x) = x^2$, (verify)

Ex. 6 : Express $m(x) = \dfrac{1}{x^3+7}$ in the form of $f \circ g \circ h$

→ In $m(x)$, x is cubed first then 7 is added and then its reciprocal taken. So,

$h(x) = x^3$, $g(x) = x + 7$ and $f(x) = \frac{1}{x}$, (verify)

6.2.2 Inverse functions:

Let $f : A \rightarrow B$ be one-one and onto function and $f(x) = y$ for $x \in A$. The inverse function

$f^{-1} : B \rightarrow A$ is defined as $f^{-1}(y) = x$ if $f(x) = y$

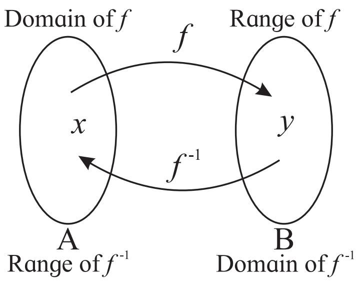


Fig. 6.36

Note:

1) As f is one-one and onto every element $y \in B$ has a unique element $x \in A$ such that $y = f(x)$.

2) If f and g are one-one and onto functions such that $f[g(x)] = x$ for every $x \in$ Domain of g and $g[f(x)] = x$ for every $x \in$ Domain of f, then g is called inverse of function f. Function g is denoted by f^{-1} (read as f inverse).
i.e. $f[g(x)] = g[f(x)] = x$ then $g = f^{-1}$ which Moreover this means $f[f^{-1}(x)] = f^{-1}[f(x)] = x$

3) $f^{-1}(x) \neq [f(x)]^{-1}$, because $[f(x)]^{-1} = \frac{1}{f(x)}$
$[f(x)]^{-1}$ is reciprocal of function $f(x)$ where as $f^{-1}(x)$ is the inverse function of $f(x)$.

e.g. If f is one-one onto function with $f(3) = 7$ then $f^{-1}(7) = 3$.

Ex. 7 : If f is one-one onto function with $f(x) = 9 - 5x$, find $f^{-1}(-1)$.

Soln. : $\rightarrow$ Let $f^{-1}(-1) = m$, then $-1 = f(m)$

Therefore,

$$-1 = 9 - 5m$$
$$5m = 9 + 1$$
$$5m = 10$$
$$m = 2$$

That is $f(2) = -1$, so $f^{-1}(-1) = 2$.

Ex. 8 : Verify that $f(x) = \frac{x-5}{8}$ and $g(x) = 8x + 5$ are inverse functions of each other.

Solution : As $f(x) = \frac{x-5}{8}$, replace x in $f(x)$ with $g(x)$

$$f[g(x)] = \frac{g(x)-5}{8} = \frac{8x+5-5}{8} = \frac{8x}{8} = x$$

and $g(x) = 8x + 5$, replace x in $g(x)$ with $f(x)$

$$g[f(x)] = 8f(x) + 5 = 8\left[\frac{x-5}{8}\right] + 5 = x - 5 + 5$$
$= x$

As $f[g(x)] = x$ and $g[f(x)] = x$, f and g are inverse functions of each other.

Ex. 9 : Determine whether the function

$f(x) = \frac{2x+1}{x-3}$ has inverse, if it exists find it.

Solution : f^{-1} exists only if f is one-one and onto.

Consider $f(x_1) = f(x_2)$,

Therefore,

$$\frac{2x_1+1}{x_1-3} = \frac{2x_2+1}{x_2-3}$$
$$(2x_1+1)(x_2-3) = (2x_2+1)(x_1-3)$$
$$2x_1x_2 - 6x_1 + x_2 - 3 = 2x_1x_2 - 6x_2 + x_1 - 3$$
$$-6x_1 + x_2 = -6x_2 + x_1$$
$$6x_1 + x_2 = 6x_2 + x_1$$
$$7x_2 = 7x_1$$
$$x_2 = x_1$$

Hence, f is one-one function.

Let $f(x) = y$, so $y = \frac{2x+1}{x-3}$

Express x as function of y, as follows

$$y = \frac{2x+1}{x-3}$$
$$y(x-3) = 2x + 1$$

$$xy - 3y = 2x + 1$$

$$xy - 2x = 3y + 1$$

$$x(y-2) = 3y + 1$$

$$\therefore \quad x = \frac{3y+1}{y-2} \text{ for } y \neq 2.$$

Thus for any $y \neq 2$,

we have x such that $f(x) = y$

f^{-1} is well defined on R - {2}

If $f(x) = 2$ i.e. $2x + 1 = 2(x - 3)$

i.e. $2x + 1 = 2x - 6$ i.e. $1 = -6$

Which is contradiction.

$2 \notin$ Range of f.

Here the range of $f(x)$ is R − {2}.

x is defined for all y in the range.

Therefore $f(x)$ is onto function.

As f is one-one and onto, so f^{-1} exists.

As $f(x) = y$, so $f^{-1}(y) = x$

Therefore, $f^{-1}(y) = \dfrac{3y+1}{y-2}$

Replace x by y, to get

$$f^{-1}(x) = \frac{3x+1}{x-2}.$$

6.2.3 Piecewise Defined Functions:

A function defined by two or more equations on different parts of the given domain is called piecewise defind function.

e.g.: If $f(x) = \begin{cases} x+1 & \text{if } x<1 \\ 4-x & \text{if } x\geq 1 \end{cases}$

Here $f(3) = 4 - 3 = 1$ as $3 > 1$,

whereas $f(-2) = -2 + 1 = -1$ as $-2 < 1$ and

$f(1) = 4 - 1 = 3$.

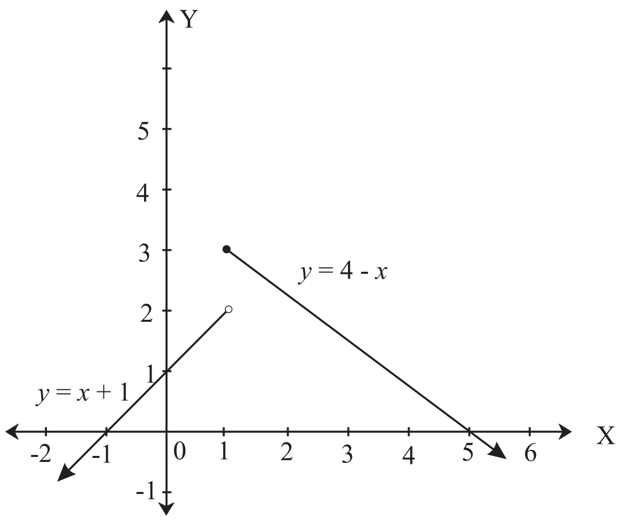


Fig. 6.37

As (1,3) lies on line $y = 4 - x$, so it is shown by small black disc on that line. (1,2) is shown by small white disc on the line $y = x + 1$, because it is not on the line.

1) Signum function :

Definition: $f(x) = \text{sgn}(x)$ is a piecewise function

$$f(x) = \begin{cases} -1 & \text{if } x<0 \\ 0 & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$$

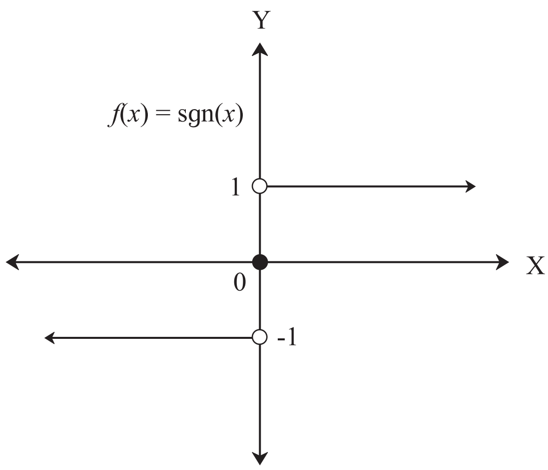


Fig. 6.38

Domain: R and **Range:** {−1, 0, 1}

Properties:

1) For $x > 0$, the graph is line $y = 1$ and for $x < 0$, the graph is line $y = -1$.

2) For $f(0) = 0$, so point (0,0) is shown by black disc, whereas points (0,−1) and (0,1) are shown by white discs.

Absolute value function (Modulus function):

Definition: $f(x) = |x|$, is a piece wise function

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

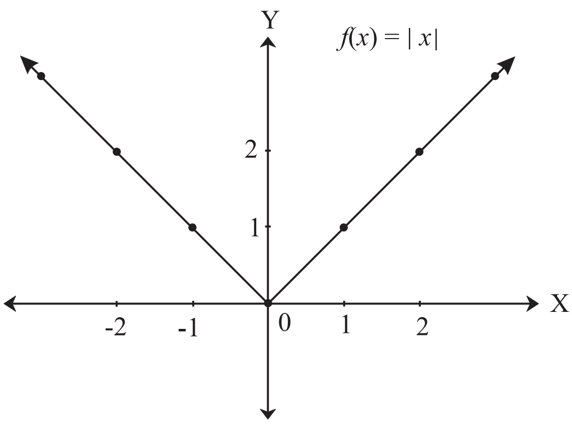


Fig. 6.39

Domain : R or $(-\infty,\infty)$ and **Range :** $[0,\infty)$

Properties:

1) Graph of $f(x) = |x|$ is union of line $y = x$ from quadrant I with the line $y = -x$ from quadrant II. As origin marks the change of directions of the two lines, we call it a critical point.

2) Graph is symmetric about y-axis .

3) Graph of $f(x) = |x-3|$ is the graph of $|x|$ shifted 3 units right and the critical point is (3,0).

4) $f(x) = |x|$, represents the distance of x from origin.

5) If $|x| = m$, then it represents every x whose distance from origin is m, that is $x = +m$ or $x = -m$.

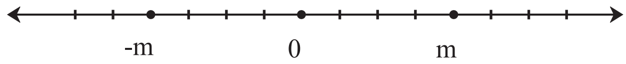


Fig. 6.40

6) If $|x| < m$, then it represents every x whose distance from origin is less than m, $0 \leq x < m$ and $0 \geq x > -m$ That is $-m < x < m$. In interval notation $x \in (-m, m)$

Fig. 6.41

7) If $|x| \geq m$, then it represents every x whose distance from origin is greater than or equal to m, so, $x \geq m$ and $x \leq -m$. In interval notation $x \in (-\infty, m] \cup [m, \infty)$

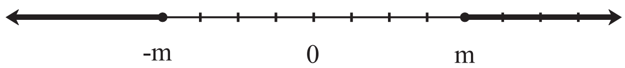


Fig. 6.42

8) If $m < |x| < n$, then it represents all x whose distance from origin is greater than m but less than n . That is $x \in (-n,-m) \cup (m, n)$.

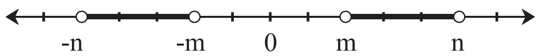


Fig. 6.43

9) Triangle inequality $|x + y| \leq |x| + |y|$.

Verify by taking different values for x and y (positive or negative).

10) $|x|$ can also be defined as $|x| = \sqrt{x^2}$ $= \max\{x, -x\}$.

Ex. 10 : Solve $|4x - 5| \leq 3$.

Solution : If $|x| \leq m$, then $-m \leq x \leq m$

Therefore

$$-3 \leq 4x - 5 \leq 3$$

$$-3 + 5 \leq 4x \leq 3 + 5$$

$$2 \leq 4x \leq 8$$

$$\frac{2}{4} \leq x \leq \frac{8}{4}$$

$$\frac{1}{2} \leq x \leq 2$$

Ex. 11 : Find the domain of $\dfrac{1}{\sqrt{||x|-1|-3}}$

Solution : As function is defined for $||x|-1|-3>0$

Therefore $||x|-1|>3$

So $|x|-1>3$ or $|x|-1<-3$

That is

$|x|>3+1$ or $|x|<-3+1$

$|x|>4$ or $|x|<-2$

But $|x|<-2$ is not possible as $|x|>0$ always

So $-4<x<4$, $x \in (-4, 4)$.

Ex. 12 : Solve $|x-1|+|x+2|=8$.

Solution : Let $f(x)=|x-1|+|x+2|$

Here the critical points are at $x=1$ and $x=-2$.

They divide number line into 3 parts, as follows.

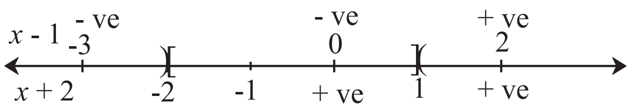


Fig. 6.44

Region	Test Value	Sign	$f(x)$
I $x<-2$	-3	$(x-1)<0$, $(x+2)<0$	$-(x-1)-(x+2)$ $=-2x-1$
II $-2\le x\le 1$	0	$(x-1)<0$, $(x+2)>0$	$-(x-1)+(x+2)$ $=3$
III $x>1$	2	$(x-1)>0$ $(x+2)>0$	$(x-1)+(x+2)$ $=2x+1$

As $f(x)=8$

From I, $-2x-1=8$ $\therefore -2x=9$ $\therefore x=-\dfrac{9}{2}$.

From II, $3=8$, which is impossible, hence there is no solution in this region.

From III, $2x+1=8$ $\therefore 2x=7$ $\therefore x=\dfrac{7}{2}$.

Solutions are $x=-\dfrac{9}{2}$ and $x=\dfrac{7}{2}$.

3) Greatest Integer Function (Step Function):

Definition: For every real x , $f(x)=[x]=$ The greatest integer less than or equal to x . $[x]$ is also called as floor function and represented by $\lfloor x \rfloor$.

Illustrations:

1) $f(5.7)=[5.7]=$ greatest integer less than or equal to 5.7

Integers less than or equal to 5.7 are 5, 4, 3, 2 of which 5 is the greatest.

2) $f(-6.3)=[-6.3]=$ greatest integer less than or equal to -6.3.

Integers less than or equal to -6.3 are $-10, -9, -8, -7$ of which -7 is the greatest.

$\therefore [-6.3]=-7$

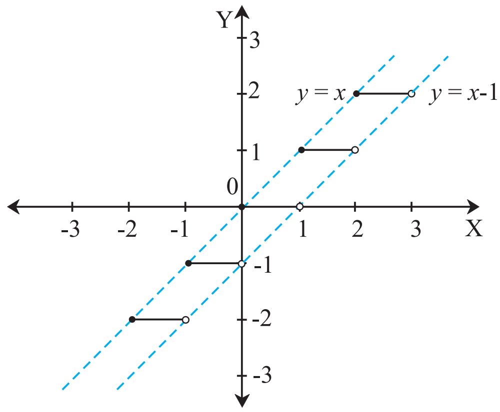
3) $f(2)=[2]=$ greatest integer less than or equal to $2=2$.

4) $[\pi]=3$ 5) $[e]=2$

The function can be defined piece-wise as follows

$f(x)=n$, if $n\le x<n+1$ or $x\in[n, n+1)$, $n\in I$

$$f(x)=\begin{cases}-2 & \text{if } -2\le x<-1 & \text{or } x\in[-2,-1)\\ -1 & \text{if } -1\le x<0 & \text{or } x\in[-1,0)\\ 0 & \text{if } 0\le x<1 & \text{or } x\in[0,1)\\ 1 & \text{if } 1\le x<2 & \text{or } x\in[1,2)\\ 2 & \text{if } 2\le x<3 & \text{or } x\in[2,3)\end{cases}$$



Graph of $f(x)=[x]$

Fig. 6.45

Domain = R and Range = I (Set of integers)

Properties:

1) If $x \in [2,3)$, $f(x) = 2$ shown by horizontal line. At exactly $x = 2$, $f(2) = 2$, $2 \in [2,3)$ hence shown by black disc, whereas $3 \notin [2,3)$ hence shown by white disc.

2) Graph of $y = [x]$ lies in the region bounded by lines $y = x$ and $y = x-1$. So $x-1 \leq [x] < x$

3) $[x] + [-x] = \begin{cases} 0 & \text{if } x \in \mathrm{I} \\ -1 & \text{if } x \notin \mathrm{I} \end{cases}$

Ex. $[3.4] + [-3.4] = 3 + (-4) = -1$ where $3.4 \notin \mathrm{I}$

$[5] + [-5] = 5 + (-5) = 0$ where $5 \in \mathrm{I}$

4) $[x+n] = [x] + n$, where $n \in \mathrm{I}$

Ex. $[4.5 + 7] = [11.5] = 11$ and

$[4.5] + 7 = 4 + 7 = 11$

4) Fractional part function:

Definition: For every real x, $f(x) = \{x\}$ is defined as $\{x\} = x - [x]$

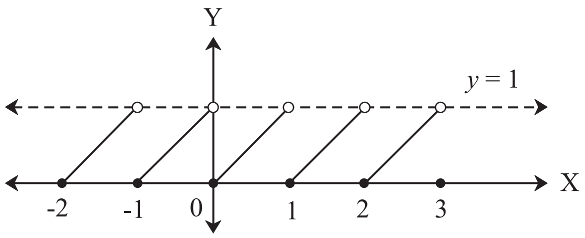
Illustrations:

$f(4.8) = \{4.8\} = 4.8 - [4.8] = 4.8 - 4 = 0.8$

$f(-7.1) = \{-7.1\} = -7.1 - [-7.1]$

$= -7.1 - (-8) = -7.1+8 = 0.9$

$f(8) = \{8\} = 8 - [8] = 8 - 8 = 0$



Graph of $f(x) = \{x\}$

Fig. 6.46

Domain = R and Range = [0,1)

Properties:

1) If $x \in [0,1]$, $f(x) = \{x\} \in [0,1)$ shown by slant line $y = x$. At $x = 0$, $f(0) = 0$, $0 \in [0,1)$ hence shown by black disc, whereas at $x = 1$, $f(1) = 1$, $1 \notin [0,1)$ hence shown by white disc.

2) Graph of $y = \{x\}$ lies in the region bounded by $y = 0$ and $y = 1$. So $0 \leq \{x\} < 1$

3) $\{x\} + \{-x\} = \begin{cases} 0 & \text{if } x \in \mathrm{I} \\ 1 & \text{if } x \notin \mathrm{I} \end{cases}$

Ex. 13: $\{5.2\}+\{-5.2\} = 0.2+0.8 = 1$ where $5.2 \in 1$

$\{7\} + \{-7\} = 0 + (0) = 0$ where $7 \in \mathrm{I}$

4) $\{x \pm n\} = \{x\}$, where $n \in \mathrm{I}$

Ex. 14 : $\{2.8+5\} = \{7.8\} = 0.8$ and $\{2.8\} = 0.8$

$\{2.8 - 5\} = \{-2.2\} = -2.2 - (-2.2) = -2.2 - (-3)$

$= 0.8$ ($\because$ $\{x\} = x - [x]$)

Ex. 15 : If $\{x\}$ and $[x]$ are the fractional part function and greatest integer function of x respectively. Solve for x, if $\{x + 1\} + 2x = 4[x + 1] - 6$.

Solution : $\{x + 1\} + 2x = 4\,[x + 1] - 6$

Since $\{x + n\} = \{x\}$ and $[x + n] = [x] + n$, for $n \in I$, also $x = [x] + \{x\}$

$\therefore$ $\{x\} + 2(\{x\} + [x]) = 4([x] + 1) - 6$

$\therefore$ $\{x\} + 2\{x\} + 2[x] = 4[x] + 4 - 6$

$\therefore$ $3\{x\} = 4[x] - 2[x] - 2$

$\therefore$ $3\{x\} = 2[x] - 2$ ... (I)

Since $0 \leq \{x\} < 1$

$\therefore$ $0 \leq 3\{x\} < 3$

$\therefore$ $0 \leq 2\,[x] - 2 < 3$ ($\because$ from I)

$\therefore$ $0 + 2 \leq 2\,[x] < 3 + 2$

$\therefore$ $2 \leq 2\,[x] < 5$

$\therefore$ $\frac{2}{2} \leq [x] < \frac{5}{2}$

$\therefore$ $1 \leq [x] < 2.5$

But as $[x]$ takes only integer values

$[x] = 1, 2$ since $[x] = 1 \Rightarrow 1 \le x < 2$ and $[x] = 2 \Rightarrow 2 \le x < 3$

Therefore $x \in [1,3)$

Note:

1)

Property	$f(x)$
$f(x+y) = f(x) + f(y)$	kx
$f(x+y) = f(x)\,f(y)$	a^{kx}
$f(xy) = f(x)\,f(y)$	x^n
$f(xy) = f(x) + f(y)$	$\log x$

2) If $n(A) = m$ and $n(B) = n$ then

(a) number of functions from A and B is n^m (b) for $m \le n$, number of one-one functions is $\dfrac{n!}{(n-m)!}$

(c) for $m > n$, number of one-one functions is 0

(d) for $m \ge n$, number of onto functions are $n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - {}^nC_3(n-3)^m + \ldots + (-1)^{n-1}\,{}^nC_{n-1}$

(e) for $m < n$, number of onto functions are 0.

(f) number of constant fuctions is m.

3) Characteristic & Mantissa of Common Logarithm $\log_{10} x$:

As $x = [x] + \{x\}$

$\log_{10} x = [\log_{10} x] + \{\log_{10} x\}$

Where, integral part $[\log_{10} x]$ is called Characteristic & fractional part $\{\log_{10} x\}$ is called Mantissa.

Illustration : For $\log_{10} 23$,

$\log_{10} 10 < \log_{10} 23 < \log_{10} 100$

$\log_{10} 10 < \log_{10} 23 < \log_{10} 10^2$

$\log_{10} 10 < \log_{10} 23 < 2\log_{10} 10$

$1 < \log_{10} 23 < 2 \quad (\because \log_{10}^{10} = 1)$

Then $[\log_{10} 23] = 1$, hence Characteristic of $\log_{10} 23$ is 1.

The characteristic of the logarithm of a number N, with 'm' digits in its integral part is 'm–1'.

Ex. 16 : Given that $\log_{10} 2 = 0.3010$, find the number of digits in the number 20^{10}.

Solution : Let $x = 20^{10}$, taking $\log_{10}$ on either sides, we get

$$\begin{aligned}\log_{10} x &= \log_{10}(20^{10}) = 10\log_{10}20\\ &= 10\log_{10}(2\times 10) = 10\{\log_{10}2 + \log_{10}10\}\\ &= 10\{\log_{10}2 + 1\} = 10\{0.3010 + 1\}\\ &= 10\,(1.3010) = 13.010\end{aligned}$$

That is characteristic of x is 13.

So number of digits in x is $13 + 1 = 14$

EXERCISE 6.2

1) If $f(x) = 3x + 5$, $g(x) = 6x - 1$, then find

(a) $(f+g)(x)$ (b) $(f-g)(2)$

(c) $(fg)(3)$ (d) $(f/g)(x)$ and its domain.

2) Let $f: \{2,4,5\} \to \{2,3,6\}$ and $g: \{2,3,6\} \to \{2,4\}$ be given by $f = \{(2,3), (4,6), (5,2)\}$ and $g = \{(2,4), (3,4), (6,2)\}$. Write down $g \circ f$

3) If $f(x) = 2x^2 + 3$, $g(x) = 5x - 2$, then find

(a) $f \circ g$ (b) $g \circ f$

(c) $f \circ f$ (d) $g \circ g$

4) Verify that f and g are inverse functions of each other, where

(a) $f(x) = \dfrac{x-7}{4}$, $g(x) = 4x + 7$

(b) $f(x) = x^3 + 4$, $g(x) = \sqrt[3]{x-4}$

(c) $f(x) = \dfrac{x+3}{x-2}$, $g(x) = \dfrac{2x+3}{x-1}$

5) Check if the following functions have an inverse function. If yes, find the inverse function.

(a) $f(x) = 5x^2$ (b) $f(x) = 8$

(c) $f(x) = \dfrac{6x-7}{3}$ (d) $f(x) = \sqrt{4x+5}$

(e) $f(x) = 9x^3+8$

(f) $f(x) = \begin{cases} x+7 & x<0 \\ 8-x & x\geq 0 \end{cases}$

6) If $f(x) = \begin{cases} x^2+3, & x\leq 2 \\ 5x+7, & x>2 \end{cases}$, then find

(a) $f(3)$ (b) $f(2)$ (c) $f(0)$

7) If $f(x) = \begin{cases} 4x-2, & x\leq -3 \\ 5, & -3<x<3 \\ x^2, & x\geq 3 \end{cases}$, then find

(a) $f(-4)$ (b) $f(-3)$

(c) $f(1)$ (d) $f(5)$

8) If $f(x) = 2|x| + 3x$, then find

(a) $f(2)$ (b) $f(-5)$

9) If $f(x) = 4[x] - 3$, where $[x]$ is greatest integer function of x, then find

(a) $f(7.2)$ (b) $f(0.5)$

(c) $f\left(-\dfrac{5}{2}\right)$ (d) $f(2\pi)$, where $\pi = 3.14$

10) If $f(x) = 2\{x\} + 5x$, where $\{x\}$ is fractional part function of x, then find

(a) $f(-1)$ (b) $f\left(\dfrac{1}{4}\right)$

(c) $f(-1.2)$ (d) $f(-6)$

11) Solve the following for x, where $|x|$ is modulus function, $[x]$ is greatest integer function, $[x]$ is a fractional part function.

(a) $|x+4|\geq 5$ (b) $|x-4| + |x-2| = 3$

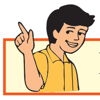
(b) $x^2 + 7|x| + 12 = 0$ (d) $|x| \leq 3$

(e) $2|x| = 5$

(f) $[x + [x + [x]]] = 9$ (g) $\{x\}>4$

(h) $\{x\} = 0$ (i) $\{x\} = 0.5$

(j) $2\{x\} = x + [x]$



Let's Remember

- If f:A $\to$ B is a function and $f(x) = y$, where $x \in$ A and $y \in$ B, then

  Domain of f is A = Set of Inputs = Set of Pre-images = Set of values of x for which $y = f(x)$ is defined = Projection of graph of $f(x)$ on X-axis.

  Range of f is f(A) = Set of Outputs = Set of Images = Set of values of y for which $y = f(x)$ is defined = Projection of graph of $f(x)$ on Y-axis.

  Co-domain of f is B.

- If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ then f is **one-one** and for every $y \in$ B, if there exists $x \in$ A such that $f(x) = y$ then f is **onto**.

- If f:A $\to$ B. g:B $\to$ C then a function $g \circ f$:A $\to$ C is a **composite function**.

- If f:A $\to$ B, then f^{-1}:B $\to$ A is **inverse function** of f.

- If f:R $\to$ R is a real valued function of real variable, the following table is formed.

Type of f	Form of f	Domain of f	Range of f
Constant function	$f(x) = k$	R	k
Identity function	$f(x) = x$	R	R
Square function	$f(x) = x^2$	R	$[0, \infty)$ or R^+
Cube function	$f(x) = x^3$	R	R
Linear function	$f(x) = ax + b$	R	R
Quadratic function	$f(x) = ax^2 + bx + c$	R	$\left(\frac{4ac - b^2}{4a}, \infty\right)$
Cubic function	$f(x) = ax^3 + bx^2 + cx + d$	R	R
Square root funtion	$f(x) = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$ or R^+
Cube root function	$f(x) = \sqrt[3]{x}$	R	R
Rational function	$f(x) = \frac{p(x)}{q(x)}$	$R - \{x \mid q(x) = 0\}$	depends on $p(x)$ and $q(x)$
Exponential function	$f(x) = a^x, a > 1$	R	$(0, \infty)$
Logarithmic function	$f(x) = \log_a x, a > 1$	$(0, \infty)$ or R^+	R
Absolute function	$f(x) = \lvert x \rvert$	R	$[0, \infty)$ or R^+
Signum function	$f(x) = \text{sgn}(x)$	R	$\{-1, 0, 1\}$
Greatest Integer function	$f(x) = [x]$	R	I (set of integers)
Fractional Part function	$f(x) = \{x\}$	R	[0,1)

MISCELLANEOUS EXERCISE 6

(I) Select the correct answer from given alternatives.

1) If $\log(5x - 9) - \log(x + 3) = \log 2$ then $x =$ ....

A) 3 B) 5 C) 2 D) 7

2) If $\log_{10}(\log_{10}(\log_{10}x)) = 0$ then $x =$

A) 1000 B) 10^{10}

C) 10 D) 0

3) Find x, if $2\log_2 x = 4$

A) 4, −4 B) 4

C) −4 D) not defined

4) The equation $\log_{x^2} 16 + \log_{2x} 64 = 3$ has,

A) one irrational solution

B) no prime solution

C) two real solutions

D) one integral solution

5) If $f(x) = \frac{1}{1-x}$, then $f(f\{f(x)\}]$ is

A) $x - 1$ B) $1 - x$ C) x D) $-x$

6) If $f : R \rightarrow R$ is defined by $f(x) = x^3$ then $f^{-1}(8)$ is euqal to :

A) {2} B) {–2. 2}

C) {–2} D) (–2. 2)

7) Let the function f be defined by $f(x) = \frac{2x+1}{1-3x}$ then $f^{-1}(x)$ is:

A) $\frac{x-1}{3x+2}$ B) $\frac{x+1}{3x-2}$

C) $\frac{2x+1}{1-3x}$ C) $\frac{3x+2}{x-1}$

8) If $f(x) = 2x^2 + bx + c$ and $f(0) = 3$ and $f(2) = 1$, then $f(1)$ is equal to

A) –2 B) 0 C) 1 D) 2

9) The domain of $\frac{1}{[x]-x}$ where $[x]$ is greatest integer function is

A) R B) Z C) R–Z D) Q - {o}

10) The domain and range of $f(x) = 2 - |x - 5|$ is

A) $R^+, (-\infty, 1]$ B) $R, (-\infty, 2]$

C) $R, (-\infty, 2)$ D) $R^+, (-\infty, 2]$

(II) Answer the following.

1) Which of the following relations are functions? If it is a function determine its domain and range.

i) {(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)}

ii) {(0, 0), (1, 1), (1, –1), (4, 2), (4, –2), (9, 3), (9, –3), (16, 4), (16, –4)}

iii) {12, 1), (3, 1), (5, 2)}

2) Find whether following functions are one-one.

i) f:R→R defined by $f(x) = x^2+5$

ii) f: R–{3}→R defined by $f(x) = \frac{5x+7}{x-3}$ for $x \in$ R–{3}

3) Find whether following functions are onto or not.

i) f: Z→Z defined by $f(x) = 6x-7$ for all $x \in Z$

ii) f: R→R defined by $f(x) = x^2+3$ for all $x \in R$

4) Let f: R→R be a function defined by $f(x) = 5x^3-8$ for all $x \in R$, show that f is one-one and onto. Hence find f^{-1}.

5) A function f: R→R defined by $f(x) = \frac{3x}{5} + 2$, $x \in R$. Show that f is one-one and onto. Hence find f^{-1}.

6) A function f is defined as $f(x) = 4x+5$, for $-4 \le x < 0$. Find the values of $f(-1)$, $f(-2)$, $f(0)$, if they exist.

7) A function f is defined as : $f(x) = 5-x$ for $0 \le x \le 4$. Find the value of x such that (i) $f(x) = 3$ (ii) $f(x) = 5$

8) If $f(x) = 3x^4 - 5x^2 + 7$ find $f(x-1)$.

9) If $f(x) = 3x +$ a and $f(1) = 7$ find a and $f(4)$.

10) If $f(x) = ax^2 + bx + 2$ and $f(1) = 3$, $f(4) = 42$, find a and b.

11) Find composite of f and g

i) f = {(1, 3), (2, 4), (3, 5), (4, 6)}
g = {(3, 6), (4, 8), (5, 10), (6, 12)}

ii) f = {(1, 1), (2, 4), (3, 4), (4, 3)}
g = {(1, 1), (3, 27), (4, 64)}

12) Find fog and gof

i) $f(x) = x^2 + 5$, $g(x) = x-8$

ii) $f(x) = 3x - 2$, $g(x) = x^2$

iii) $f(x) = 256x^4$, $g(x) = \sqrt{x}$

13) If $f(x) = \frac{2x-1}{5x-2}$, $x \neq \frac{5}{2}$

Show that $(fof)(x) = x$.

14) If $f(x) = \dfrac{x+3}{4x-5}$, $g(x) = \dfrac{3+5x}{4x-1}$ then show that $(fog)(x) = x$.

15) Let $f: R - \{2\} \rightarrow R$ be defined by $f(x) = \dfrac{x^2-4}{x-2}$ and $g : R \rightarrow R$ be defined by $g(x) = x + 2$. Ex whether $f = g$ or not.

16) Let $f: R \rightarrow R$ be given by $f(x) = x + 5$ for all $x \in R$. Draw its graph.

17) Let $f: R \rightarrow R$ be given by $f(x) = x^3 + 1$ for all $x \in R$. Draw its graph.

18) For any base show that

$\log(1+2+3) = \log 1 + \log 2 + \log 3$.

19) Find x, if $x = 3^{3\log_3 2}$

20) Show that,

$$\log|\sqrt{x^2+1}+x| + \log|\sqrt{x^2+1}-x| = 0$$

21) Show that, $\log\dfrac{a^2}{bc} + \log\dfrac{b^2}{ca} + \log\dfrac{c^2}{ab} = 0$

22) Simplify, $\log(\log x^4) - \log(\log x)$.

23) Simplify

$$\log_{10}\frac{28}{45} - \log_{10}\frac{35}{324} + \log_{10}\frac{325}{432} - \log_{10}\frac{13}{15}$$

24) If $\log\left(\dfrac{a+b}{2}\right) = \dfrac{1}{2}(\log a + \log b)$, then show that a=b

25) If $b^2=ac$. prove that, $\log a + \log c = 2\log b$

26) Solve for x, $\log_x(8x-3) - \log_x 4 = 2$

27) If $a^2 + b^2 = 7ab$, show that,

$$\log\left(\frac{a+b}{2}\right) = \frac{1}{2}\log a + \frac{1}{2}\log b$$

28) If $\log\left(\dfrac{x-y}{5}\right) = \dfrac{1}{2}\log x + \dfrac{1}{2}\log y$, show that $x^2 + y^2 = 27xy$.

29) If $\log_3[\log_2(\log_3 x)] = 1$, show that $x = 6561$.

30) If $f(x) = \log(1-x)$, $0 \le x < 1$ show that

$$f\left(\frac{1}{1+x}\right) = f(1-x) - f(-x)$$

31) Without using log tables, prove that

$$\frac{2}{5} < \log_{10}3 < \frac{1}{2}$$

32) Show that

$$7\log\left(\frac{15}{16}\right) + 6\log\left(\frac{8}{3}\right) + 5\log\left(\frac{2}{5}\right) + \log\left(\frac{32}{25}\right)$$

$= \log 3$

33) Solve : $\sqrt{\log_2 x^4} + 4\log_4\sqrt{\dfrac{2}{x}} = 2$

34) Find value of $\dfrac{3+\log_{10}343}{2+\frac{1}{2}\log_{10}\left(\frac{49}{4}\right)+\frac{1}{2}\log_{10}\left(\frac{1}{25}\right)}$

35) If $\dfrac{\log a}{x+y-2z} = \dfrac{\log b}{x+y-2x} = \dfrac{\log c}{x+y-2y}$, show that $abc = 1$.

36) Show that, $\log_y x^3 . \log_z y^4 . \log_x z^5 = 60$

37) If $\dfrac{\log_2 a}{4} = \dfrac{\log_2 b}{6} = \dfrac{\log_2 c}{3k}$ and $a^3b^2c = 1$ find the value of k.

38) If $a^2 = b^3 = c^4 d^5$, show that $\log_a bcd = \dfrac{47}{30}$.

39) Solve the following for x, where $|x|$ is modulus function, $[x]$ is greatest interger function, $\{x\}$ is a fractional part function.

a) $1 < |x-1| < 4$ c) $|x^2 - x - 6| = x + 2$

c) $|x^2 - 9| + |x^2 - 4| = 5$

d) $-2 < [x] \le 7$ e) $2[2x-5] - 1 = 7$

f) $[x^2] - 5[x] + 6 = 0$

g) $[x-2] + [x+2] + \{x\} = 0$

h) $\left[\dfrac{x}{2}\right] + \left[\dfrac{x}{3}\right] = \dfrac{5x}{6}$

40) Find the domain of the following functions.

a) $f(x)=\dfrac{x^2+4x+4}{x^2+x-6}$

b) $f(x)=\sqrt{x-3}+\dfrac{1}{\log(5-x)}$

c) $f(x)=\sqrt{1-\sqrt{1-\sqrt{1-x^2}}}$

d) $f(x)=x!$

e) $f(x)={}^{5-x}P_{x-1}$

f) $f(x)=\sqrt{x-x^2}+\sqrt{5-x}$

g) $f(x)=\sqrt{\log(x^2-6x+6)}$

41) Find the range of the following functions.

a) $f(x)=|x-5|$ b) $f(x)=\dfrac{x}{9+x^2}$

c) $f(x)=\dfrac{1}{1+\sqrt{x}}$ d) $f(x)=[x]-x$

e) $f(x)=1+2^x+4^x$

42) Find $(f\circ g)(x)$ and $(g\circ f)(x)$

a) $f(x)=e^x$, $g(x)=\log x$

b) $f(x)=\dfrac{x}{x+1}$, $g(x)=\dfrac{x}{1-x}$

43) Find $f(x)$ if

a) $g(x) = x^2 + x - 2$ and $(g\circ f)(x) = 4x^2-10x+4$

(b) $g(x)=1+\sqrt{x}$ and $f[g(x)]=3+2\sqrt{x}+x$.

44) Find $(f\circ f)(x)$ if

(a) $f(x)=\dfrac{x}{\sqrt{1+x^2}}$

(b) $f(x)=\dfrac{2x+1}{3x-2}$

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