8 CONTINUITY

Let's Study

- Continuity of a function at a point.
- Continuity of a function over an interval.
- Intermediate value theorem.

Let's Recall

- Different types of functions.
- Limits of Algebraic, Trigonometric, Exponential and Logarithmic functions.
- Left hand and Right hand limits of functions.

8.1 CONTINUOUSANDDISCONTINUOUS FUNCTIONS

The dictionary meaning of the word continuity **is the unbroken and consistent existence over a period of time.** The intuitive idea of continuity is manifested in the following examples.

(i) An unbroken road between two cities.

(ii) Flow of river water.

(iii) Railway tracks.

(iv) The changing temperature of a city during a day.

In winter the temperature of Pune rises from 14^0C at night to 29^0C in the afternoon. This change in the temp is continuous and all the values between 14 and 29 are taken during 12 hours. An activity that takes place gradually, without interruption or abrupt change is called a continuous process. There are no jumps, breaks, gaps or holes in the graph of the function.

8.1.1 CONTINUITYOFAFUNCTIONATA POINT

We are going to study continuity of functions of real variable so the domain will be an interval in R. Before we consider a formal definition of a function to be continuous at a point, let's consider various functions that fail to meet our notion of continuity. The functions are indicated by graphs where $y = f(x)$

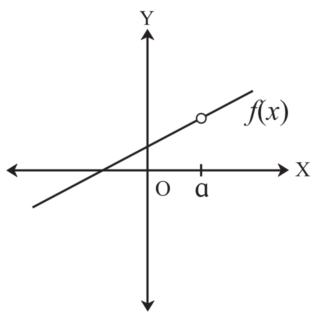

Fig. 8.1

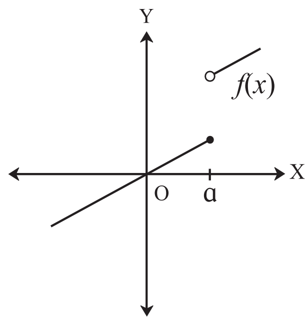

Fig. 8.2

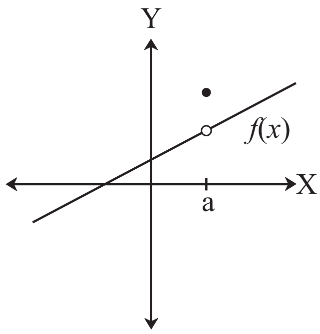

Fig. 8.3

The function in figure 8.1 has a hole at $x = a$. In fact $f(x)$ is not defined at $x = a$.

The function in figure 8.2 has a break at $x = a$.

For the function in figure 8.3, $f(a)$ is not in the continuous line.

8.1.2 DEFINITION OF CONTINUITY

A function $f(x)$ is said to be continuous at a point $x = a$, if the following three conditions are satisfied:

i. f is defined at every point on an open interval containing a.

ii. $\lim\limits_{x \to a} f(x)$ exists

iii. $\lim\limits_{x \to a} f(x) = f(a)$.

Among the three graphs given above, decide which conditions of continuity are not satisfied.

The condition (iii) can be reformulated and the continuity of $f(x)$ at $x = a$, can be restated as follows :

A function $f(x)$ is said to be continuous at a point $x = a$ if it is defined in some neighborhood of 'a' and if

$$\lim_{h \to 0} [f(a + h) - f(a)] = 0.$$

Illustration 1. Let $f(x) = |x|$ be defined on R.

$$f(x) = -x \ , \ \text{for } x < 0$$
$$= x \ , \ \text{for } x \geq 0$$

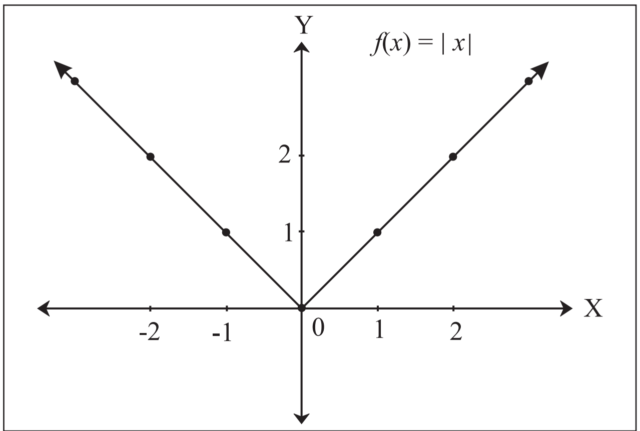

Fig. 8.4

Consider, $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0} (-x) = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (x) = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0$$

Hence $f(x)$ is continuous at $x = 0$.

Illustration 2 : Consider $f(x) = x^2$ and let us discuss the continuity of f at $x = 2$.

$$f(x) = x^2$$

$$\therefore \ f(2) = 2^2 = 4$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x^2) = 2^2 = 4$$

$$\therefore \lim_{x \to 2} f(x) = f(2) = 4$$

$\therefore$ The function $f(x)$ is continuous at $x = 2$.

Observe that $f(x) = x^3, x^4$, ... etc. are continuous at every point. It follows that all polynomials are continuous functions of x.

There are some functions, which are defined in two different ways on either side of a point. In such cases we have to consider the limits of function from left as well as right of that point.

8.1.3 CONTINUITY FROM THE RIGHT AND FROM THE LEFT

A function $f(x)$ is said to be continuous from the right at $x = a$ if $\lim\limits_{x \to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be continuous from the left at $x = a$ if $\lim\limits_{x \to a^-} f(x) = f(a)$.

If a function is continuous on the right and also on the left of a then it is continuous at a because

$$\lim_{x \to a^+} f(x) = f(a) = \lim_{x \to a^-} f(x) .$$

Illustration 3: Consider the function $f(x) = \lfloor x \rfloor$ in the interval [2, 4).

Note : $\lfloor x \rfloor$ is the greatest integer function or floor function.

Solution :

$$f(x) = \lfloor x \rfloor, \quad \text{for } x \in [2, 4)$$

that is $f(x) = 2$, for $x \in [2, 3)$

$= 3$, for $x \in [3, 4)$

The graph of which is as shown in figure 8.5

Test of continuity at $x = 3$.

For $x = 3$, $f(3) = 3$

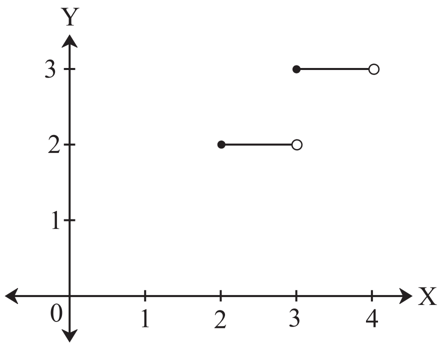

Fig. 8.5

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \lfloor x \rfloor = \lim_{x\to 3^-} (2) = 2, \text{ and}$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \lfloor x \rfloor = \lim_{x\to 3^+} (3) = 3$$

$$\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$$

$\therefore$ $f(x)$ is discontinuous at $x = 3$.

Illustration 4 :

Consider $f(x) = x^2 + \dfrac{3}{2}$ for $0 \le x \le 3$

$= 5x - 4.5$ for $3 < x \le 5$;

For $x = 3$, $f(3) = 3^2 + \dfrac{3}{2} = 10.5$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3} \left(x^2 + \frac{3}{2}\right) = 3^2 + \frac{3}{2} = 10.5$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3} (5x - 4.5) = 15 - 4.5 = 10.5$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 10.5$$

$$\therefore \lim_{x\to 3} f(x) = f(3)$$

$\therefore$ $f(x)$ is continuous at $x = 3$.

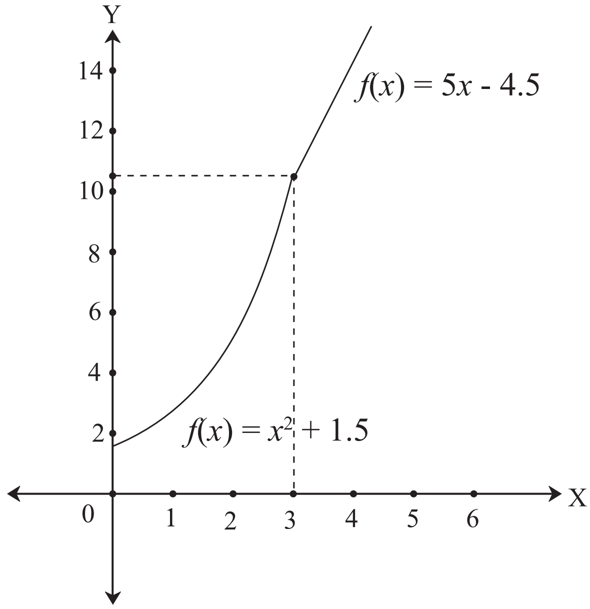

Fig. 8.6

8.1.4 Examples of Continuous Functions.

(1) Constant function, that is $f(x) = k$, is continuous at every point on R.

(2) Power functions, that is $f(x) = x^n$, with positive integral exponents are continuous at every point on R.

(3) Polynomial functions,

$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n$ are continuous at every point on R

(4) The trigonometric functions sin x and cos x are continuous at every point on R.

(5) The exponential function a^x $(a > 0)$ and logarithmic function $\log_b x$ (for $x, > 0$, and b, $b \neq 1$) are continuous on R.

(6) Rational functions are of the form $\dfrac{P(x)}{Q(x)}$, $Q(x) \neq 0$. They are continuous at every point a if $Q(a) \neq 0$.

8.1.5 PROPERTIES OF CONTINUOUS FUNCTIONS:

If the functions f and g are continuous at $x = a$, then,

1. their sum, that is $(f + g)$ is continuous at $x = a$.
2. their difference, that is $(f - g)$ or $(g - f)$ is continuous at $x = a$.
3. the constant multiple of $f(x)$, that is $k.f$, for any $k \in R$, is continuous at $x = a$.
4. their product, that is $(f.g)$ is continuous at $x = a$.
5. their quotient, that is $\dfrac{f}{g}$, if $g(a) \neq 0$, is continuous at $x = a$.
6. their composite function, $f[g(x)]$ or $g[f(x)]$, that is $fog(x)$ or $gof(x)$, is continuous at $x = a$.

8.1.6 TYPES OF DISCONTINUITIES

We have seen that discontinuities have several different types. Let us classify the types of discontinuities.

8.1.7 JUMP DISCONTINUITY

As in figure 8.2, for a function, both left-hand limit and right-hand limits may exist but they are different. So the graph "jumps" at $x = a$. The function is said to have a jump discontinuity.

> **A function $f(x)$ has a Jump Discontinuity at $x = a$ if the left hand and right-hand limits both exist but are different, that is**
>
> $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$

Illustration 5: Consider

$$f(x) = x^2 - x - 5, \text{ for } -4 \leq x < -2.$$
$$= x^3 - 4x - 3, \text{ for } -2 \leq x \leq 1.$$

For $x = -2$, $f(-2) = (-2)^3 - 4(-2) - 3 = -3$

$\lim_{x\to -2^-} f(x) = \lim_{x\to -2} (x^2 - x - 5) = 4 + 2 - 5 = 1$ and

$\lim_{x\to -2^+} f(x) = \lim_{x\to -2} (x^3 - 4x - 3) = -8 + 8 - 3 = -3$

$\therefore \lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x)$

Hence $\lim_{x\to -2^-} f(x)$ does not exist.

$\therefore$ the function $f(x)$ has a jump discontinuity.

8.1.8 REMOVABLE DISCONTINUITY

Some functions have a discontinuity at some point, but it is possible to define or redefine the function at that point to make it continuous. These types of functions are said to have a **removable discontinuity**. Let us look at the function $f(x)$ represented by the graph in Figure 8.1 or Figure 8.3. The function has a limit. However, there is a hole or gap at $x = a$. $f(x)$ is not defined at $x = a$. That can be corrected by defining $f(x)$ at $x = a$.

> **A function $f(x)$ has a discontinuity at $x = a$, and $\lim_{x\to a} f(x)$ exists, but either $f(a)$ is not defined or $\lim_{x\to a} f(x) \neq f(a)$. In such case we define or redefine $f(a)$ as $\lim_{x\to a} f(x)$. Then with new definition, the function $f(x)$ becomes continuous at $x = a$. Such a discontinuity is called a Removable discontinuity.**

If the original function is not defined at a and the new definition of f makes it continuous at a, then the new definition is called the extension of the original function.

Illustration 6:

Consider $f(x) = \dfrac{x^2+3x-10}{x^3-8}$, for $x \neq 2$.

Here $f(2)$ is not defined.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\left(\frac{x^2+3x-10}{x^3-8}\right)$$

$$= \lim_{x\to 2}\left(\frac{(x-2)(x+5)}{(x-2)(x^2+2x+4)}\right)$$

$$= \lim_{x\to 2}\left(\frac{x+5}{x^2+2x+4}\right) = \frac{2+5}{4+4+4} = \frac{7}{12}$$

$$\therefore \lim_{x\to 2}\left(\frac{x^2+3x-10}{x^3-8}\right) = \frac{7}{12}$$

Here $f(2)$ is not defined but $\lim_{x\to 2} f(x)$ exists.

Hence $f(x)$ has a removable discontinuity.

The extension of the original function is

$$f(x) = \frac{x^2+3x-10}{x^3-8} \quad \text{for } x \neq 2$$

$$= \frac{7}{12} \quad \text{for } x = 2$$

This is coninuous at $x = 2$.

8.1.9 INFINITE DISCONTINUITY

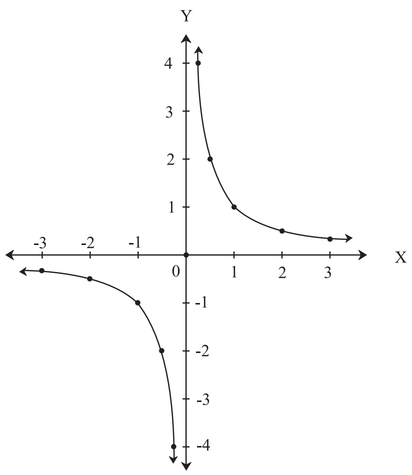

Fig. 8.7

Observe the graph of $xy = 1$. $y = f(x) = \dfrac{1}{x}$ is the function to be considered. It is easy to see that $f(x) \to \infty$ as $x \to 0^+$ and $f(x) \to -\infty$ as $x \to 0^-$. $f(0)$ is not defined. Of course, this function is discontinuous at $x = 0$.

A function $f(x)$ is said to have an infinite discontinuity at $x = a$,

if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$

Fig. 8.7 says, $f(x)$ has an infinite discontinuity.

8.1.10 CONTINUITY OVER AN INTERVAL

So far we have explored the concept of continuity of a function at a point. Now we will extend the idea of continuity on an interval.

Let (a, b) be an open interval. **If for every $x \in (a, b)$, f is continuous at x then we say that f is continuous on (a, b) .**

Consider f defined on $[a, b)$. If f is continuous on (a, b) and f is continuous to the right of a, $\lim_{x\to a^+} f(x) = f(a)$ then f is continuous on $[a, b)$

Consider f defined on $(a, b]$. If f is continuous on (a, b) and f is continuous to the left of b $\lim_{x\to a^+} f(x) = f(b)$, then f is continuous on $(a, b]$

Consider a function f continuous on the open interval (a, b). If $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exists, then we can extend the function to [a, b] so that it is continuous on [a, b].

SOLVED EXAMPLES

Ex. 1. : Discuss the continuity of the function $f(x) = |x - 3|$ at $x = 3$.

Solution : By definition of a modulus function, the given function can be rewritten as

$$f(x) = -(x-3) \quad \text{if } x < 3$$
$$= x - 3 \quad \text{if } x \geq 3$$

Now, for $x = 3$, $f(3) = 3 - 3 = 0$.

$\lim_{x \to 3^-} f(x) = -(3-3) = 0$ and

$\lim_{x \to 3^+} f(x) = 3 - 3 = 0$

so, $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 0 \Rightarrow \lim_{x \to 3} f(x) = 0$

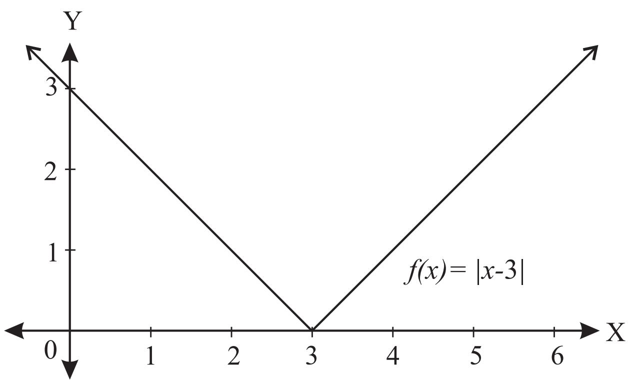

Fig. 8.8

and $\lim_{x \to 3} f(x) = f(3) = 0$

Therefore the function $f(x)$ is continuous at $x = 3$.

Ex. 2 : Determine whether the function f is continuous on the set of real numbers

where $f(x) = 3x + 1, \quad \text{for } x < 2$

$= 7, \quad \text{for } 2 \leq x < 4$

$= x^2 - 8 \quad \text{for } x \geq 4$.

If it is discontinuous, state the type of discontinuity.

Solution : The function is defined in three parts, by polynomial functions, and all polynomial functions are continuous on their respective domains. Any discontinuity, if at all it exists, would be at the points where the definition changes. That is at $x = 2$ and $x = 4$.

Let us check at $x = 2$.

$f(2) = 7$ (Given)

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3x + 1) = 3(2) + 1 = 7$.

and $\lim_{x \to 2^+} f(x) = 7$,

So $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 7 \Rightarrow \lim_{x \to 2} f(x) = 7$

Also, $\lim_{x \to 2} f(x) = f(2) = 7$

$\therefore$ $f(x)$ is continuous at $x = 2$.

Let us check the continuity at $x = 4$.

$f(4) = (4^2 - 8) = 8$

$\lim_{x \to 4^-} f(x) = \lim_{x \to 4} (7) = 7$ and

$\lim_{x \to 4^+} f(x) = \lim_{x \to 4} (x^2 - 8) = 4^2 - 8 = 8$

$\lim_{x \to 4^-} f(x) \neq \lim_{x \to 4^+} f(x)$

so $\lim_{x \to 4} f(x)$ does not exist.

Since one of the three conditions does not hold at $x = 4$, the function. Hence $f(x)$ is discontinuous at $x = 4$. Therefore the function $f(x)$ is continuous on it's domain   , except at $x = 4$. There exists a jump discontinuity at $x = 4$.

$\therefore$ f is discontinuous at $x = 4$.

Ex. 3 : Test whether the function $f(x)$ is continuous at $x = -4$, where

$$f(x) = \frac{x^2 + 16x + 48}{x + 4}, \quad \text{for } x \neq -4$$
$$= 8, \quad \text{for } x = -4.$$

Solution : $f(-4) = 8$ (defined)

$$\lim_{x \to -4} f(x) = \lim_{x \to -4} \left(\frac{x^2 + 16x + 48}{x + 4} \right)$$
$$= \lim_{x \to -4} \left(\frac{(x+4)(x+12)}{x + 4} \right)$$
$$= \lim_{x \to -4} (x + 12) \text{ .... } [\because x + 4 \neq 0]$$
$$= -4 + 12 = 8$$

$\therefore \lim_{x \to -4} f(x) = f(-4) = 8$

$\therefore$ by definition, the function $f(x)$ is continuous at $x = -4$.

Ex. 4 : Discuss the continuity of $f(x) = \sqrt{9-a^2}$, on the interval $[-3, 3]$.

Solution : The domain of f is $[-3, 3]$.

[Note, $f(x)$ is defined if $9 - x^2 \geq 0$]

Let $x = a$ be any point in the interval $(-3, 3)$ that is $a \in (-3, 3)$.

$$\lim_{x\to a} f(x) = \lim_{x\to a} \sqrt{9-x^2}$$

$$= \sqrt{9-a^2} = f(a)$$

$\therefore$ for $a = 3$, $f(3) = 0$ and for $a = -3$, $f(-3) = 0$

Now, $\lim\limits_{x\to 3^-} f(x) = f(3) = 0$

and $\lim\limits_{x\to -3^+} f(x) = f(-3) = 0$

Thus $f(x)$ is continuous at every point on $(-3, 3)$ and also continuous to the right at $x = -3$ and to the left at $x = 3$.

Hence, $f(x)$ is continuous on $[-3, 3]$.

Ex. 5 : Show that the function $f(x) = \lfloor x \rfloor$ is not continuous at $x = 0, 1$ in the interval $[-1, 2)$

Solution : $f(x) = \lfloor x \rfloor$ for $x \in [-1, 3)$

that is $f(x) = -1$ for $x \in [-1, 0)$

$f(x) = 0$ for $x \in [0, 1)$

$f(x) = 1$ for $x \in [1, 2)$

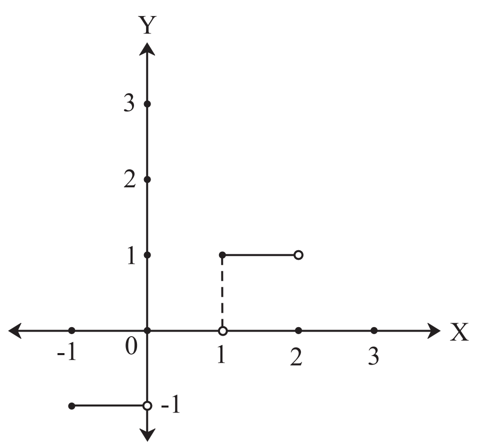

Fig. 8.9

Test of continuity of f at $x = 0$.

For $x = 0$, $f(0) = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \lfloor x \rfloor = \lim_{x\to 0^-} (-1) = -1 \text{ and}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \lfloor x \rfloor = \lim_{x\to 0^+} (0) = 0$$

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

Therefore $f(x)$ is discontinuous at $x = 0$.

Test of continuity of f at $x = 1$.

For $x = 1$, $f(1) = 1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \lfloor x \rfloor = \lim_{x\to 1^-} (0) = 0 \text{ and}$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \lfloor x \rfloor = \lim_{x\to 1^+} (1) = 1$$

$$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

Therefore $f(x)$ is discontinuous at $x = 1$.

Hence the function $f(x) = \lfloor x \rfloor$ is not continuous at $x = 0, 1$ in the interval $[-1, 2)$.

Ex. 6 : Discuss the continuity of the following function at $x = 0$, where

$$f(x) = x^2 \sin\left(\frac{1}{x}\right), \text{for } x \neq 0$$

$$= 0, \text{ for } x = 0.$$

Solution : The function $f(x)$ is defined for all $x \in \mathbb{R}$.

Let's check the continuity of $f(x)$ at $x = 0$.

Given, for $x = 0$, $f(0) = 0$.

we know that, $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ for any $x \neq 0$

Multiplying throughout by x^2 we get

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Taking limit as $x \to 0$ throughout we get,

$$\lim_{x\to 0} (-x^2) \le \lim_{x\to 0} \left[x^2 \sin\left(\frac{1}{x}\right)\right] \le \lim_{x\to 0} (x^2)$$

$$0 \le \lim_{x\to 0}\left[x^2 \sin\left(\frac{1}{x}\right)\right] \le 0$$

$\therefore$ by squeeze theorem we get,

$$\lim_{x\to 0} \left[x^2 \sin\left(\frac{1}{x}\right)\right] = 0$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \left[x^2 \sin\left(\frac{1}{x}\right)\right] = 0$$

$$\lim_{x\to 0} f(x) = f(0) = 0$$

$\therefore$ $f(x)$ is continuous at $x = 0$

Ex. 7 : Find k if $f(x)$ is continuous at $x = 0$, where

$$f(x) = \frac{xe^x + \tan x}{\sin 3x}, \quad \text{for } x \neq 0$$
$$= k, \quad \text{for } x = 0$$

Solution : Given that $f(x)$ is continuous at $x = 0$,

$$\therefore\ f(0) = \lim_{x\to 0} f(x)$$

$$k = \lim_{x\to 0} \left(\frac{xe^x + \tan x}{\sin 3x}\right)$$

$$= \lim_{x\to 0} \left(\frac{e^x + \dfrac{\tan x}{x}}{\dfrac{\sin 3x}{x}}\right)$$

$$= \frac{\lim\limits_{x\to 0}(e^x) + \lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)}{\lim\limits_{x\to 0}\left(\dfrac{\sin 3x}{3x}\right) \times 3}$$

$$= \frac{1+1}{1} \times \frac{1}{3} = \frac{2}{3} \quad \text{as } x \to 0,\ 3x \to 0$$

$$\left[\text{since } \lim_{\theta\to 0} \left(\frac{\sin\theta}{\theta}\right) = 1 = \lim_{\theta\to 0} \left(\frac{\tan\theta}{\theta}\right)\right]$$

$$\therefore\ k = \frac{2}{3}$$

Ex. 8 : If f is continuous at $x = 1$, where

$$f(x) = \frac{\sin(\pi x)}{x-1} + a, \quad \text{for } x < 1$$
$$= 2\pi, \quad \text{for } x = 1$$
$$= \frac{1+\cos(\pi x)}{\pi(1-x)^2} + b, \quad \text{for } x > 1,$$

then find the values of a and b.

Solution : Given that $f(x)$ is continuous at $x = 1$

$$\therefore\ \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) \quad \text{............ (1)}$$

Now, $\lim\limits_{x\to 1^-} f(x) = f(1)$

$$\lim_{x\to 1^-} \left(\frac{\sin(\pi x)}{x-1} + a\right) = 2\pi$$

Put $x - 1 = \text{t}$, $x = 1 + \text{t}$ as $x \to 1$, $t \to 0$

$$\lim_{t\to 0} \left(\frac{\sin \pi(1+t)}{t} + a\right) = 2\pi$$

$$\lim_{t\to 0} \left(\frac{\sin(\pi + \pi t)}{t} + a\right) = 2\pi$$

$$\lim_{t\to 0} \left(\frac{-\sin \pi t}{t} + a\right) = 2\pi$$

$$-\lim_{t\to 0} \left(\frac{\sin \pi t}{\pi t}\right) \times \pi + \lim_{t\to 0} (a) = 2\pi$$

$$-(1)\,\pi + a = 2\pi \Rightarrow a = 3\pi$$

$$\left[\lim_{\theta\to 0} \left(\frac{\sin\theta}{\theta}\right) = 1\right]$$

From (1), $\lim\limits_{x\to 1^+} f(x) = f(1)$

$$\lim_{x\to 1^+}\left(\frac{1+\cos(\pi x)}{\pi(1-x)^2}+b\right) = 2\pi$$

Put $1-x=\theta$ $\therefore$ $x=1-\theta$, as $x\to 1$, $\theta\to 0$

$$\therefore \lim_{\theta\to 0}\left(\frac{1+\cos(\pi(1-\theta))}{\pi\theta^2}+b\right) = 2\pi$$

$$\therefore \lim_{\theta\to 0}\left(\frac{1+\cos(\pi-\pi\theta)}{\pi\theta^2}+b\right) = 2\pi$$

$$\therefore \lim_{\theta\to 0}\left(\frac{1-\cos\pi\theta}{\pi\theta^2}+b\right) = 2\pi$$

$$\therefore \lim_{\theta\to 0}\left(\frac{2\sin^2\left(\frac{\pi\theta}{2}\right)}{\pi\theta^2}+b\right) = 2\pi$$

$$\frac{2}{\pi}\lim_{\theta\to 0}\left[\frac{\sin\left(\frac{\pi\theta}{2}\right)}{\frac{\pi\theta}{2}}\right]^2\left(\frac{\pi}{2}\right)^2 + \lim_{\theta\to 0}(b) = 2\pi$$

$$\frac{2}{\pi}(1)\times\frac{\pi^2}{4}+b = 2\pi \qquad \left[\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1\right]$$

$$\therefore \frac{\pi}{2}+b=2\pi \qquad \therefore b=\frac{3\pi}{2}$$

$$\therefore a=3\pi\ ,\ b=\frac{3\pi}{2}$$

Ex. 9 : Identify discontinuities for the following functions as either a jump or a removable discontinuity on R.

(1) $f(x) = \dfrac{x^2-3x-18}{x-6}$,

(2) $g(x) = 3x+1,$ for $x<3$
$\quad = 2-3x,$ for $x\geq 3$

(3) $h(x) = 13-x^2,$ for $x<5$
$\quad = 13-5x,$ for $x>5$

Solution :

(1) $f(x) = \dfrac{x^2-3x-18}{x-6}$

Here $f(x)$ is a rational function, which is continuous for all real values of x, except for $x=6$. Therefore $f(6)$ is not defined.

$$\text{Now, } \lim_{x\to 6} f(x) = \lim_{x\to 6}\left(\frac{x^2-3x-18}{x-6}\right)$$

$$= \lim_{x\to 6}\left(\frac{(x-6)(x+3)}{x-6}\right)$$

$$= \lim_{x\to 6}(x+3) \quad [\because (x-6)\neq 0]$$

$$\therefore \lim_{x\to 6} f(x) = 9$$

Here $f(6)$ is not defined but $\lim_{x\to 6} f(x)$ exists.

Hence $f(x)$ has a removable discontinuity.

(2) $g(x) = 3x+1,$ for $x<3$
$\quad = 2-3x,$ for $x\geq 3$

This function is defined by different polynomials on two intervals. So they are continuous on the open intervals $(-\infty, 3)$ and $(3, \infty)$.

We examine continuity at $x=3$.

For $x=3$, $g(3) = 2-3(3) = -7$

$$\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-}(3x+1) = 10 \text{ and}$$

$$\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+}(2-3x) = -7$$

$$\lim_{x\to 3^-} g(x) \neq \lim_{x\to 3^+} g(x) \ \therefore \lim_{x\to 3} g(x) \text{ does not exist.}$$

Hence g is not continuous at $x=3$.

The function $g(x)$ has a jump discontinuity at $x=3$.

(3) $h(x) = 13-x^2$, for $x<5$,
$\quad = 13-5x$, for $x>5$,

but $h(5)$ is not defined.

$h(x)$ is continuous at any $x < 5$ and $x > 5$

$$\lim_{x\to 5^-} h(x) = \lim_{x\to 5^-} (13 - x^2) = 13 - 25 = -12$$

and $\lim_{x\to 5^+} h(x) = \lim_{x\to 5^+} (13 - 5x) = 13 - 25 = -12$

So, $\lim_{x\to 5^-} h(x) = \lim_{x\to 5^+} h(x) \therefore \lim_{x\to 5} h(x) = -12$

But for $x = 5$, $f(x)$ is not defined.

So the function $h(x)$ has a removable discontinuity.

Note :

We have proved that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Some standard limits are stated without proof.

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$$

$$\lim_{x\to 0} (1+t)^{\frac{1}{t}} = e$$

$$\lim_{x\to 0} (1-t)^{\frac{1}{t}} = e^{-1} = \frac{1}{e}$$

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1,$$

$$\lim_{x\to 0} \frac{\log(1-x)}{x} = -1$$

These can be proved using L' Hospital's rule, or expressions in power series which will be studied at advanced stage.

Ex. 10 : Show that the function

$$f(x) = \frac{5^{\cos x} - e^{\left(\frac{\pi}{2} - x\right)}}{\cot x}, \quad \text{for } x \neq \frac{\pi}{2}$$

$$= \log 5 - e, \quad \text{for } x = \frac{\pi}{2}$$

has a removable discontinuity at $x = \frac{\pi}{2}$.

Redefine the function so that it becomes continuous at $x = \frac{\pi}{2}$.

Solution : $f(\pi/2) = \log 5 - e$

Now, $\lim_{x\to \frac{\pi}{2}} f(x) = \lim_{x\to \frac{\pi}{2}} \left(\frac{5^{\cos x} - e^{\left(\frac{\pi}{2} - x\right)}}{\cot x}\right)$

Let $\frac{\pi}{2} - x = t$, $x = \frac{\pi}{2} - t$ as $x \to \frac{\pi}{2}$, $t \to 0$

$$\lim_{x\to \frac{\pi}{2}} f(x) = \lim_{t\to 0} \left(\frac{5^{\cos\left(\frac{\pi}{2} - t\right)} - e^t}{\cot\left(\frac{\pi}{2} - t\right)}\right)$$

$$= \lim_{t\to 0} \left(\frac{5^{\sin t} - e^t}{\tan t}\right)$$

$$= \lim_{t\to 0} \left(\frac{5^{\sin t} - 1 - e^t + 1}{\tan t}\right)$$

$$= \lim_{t\to 0} \left(\frac{(5^{\sin t} - 1) - (e^t - 1)}{\tan t}\right)$$

$$= \lim_{t\to 0} \left(\frac{\frac{5^{\sin t} - 1}{t} - \frac{e^t - 1}{t}}{\frac{\tan t}{t}}\right)$$

[As t→0, t ≠ 0]

$$= \lim_{t\to 0} \left(\frac{\frac{\sin t}{t} \times \frac{5^{\sin t} - 1}{\sin t} - \frac{e^t - 1}{t}}{\frac{\tan t}{t}}\right)$$

[sin t ≠ 0]

$$= \frac{(1).(\log 5) - 1}{1} \quad \left[\begin{array}{l} \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1 = \lim_{\theta\to 0}\left(\frac{\tan\theta}{\theta}\right) \\ \lim_{x\to 0}\left(\frac{e^x - 1}{x}\right) = 1, \lim_{x\to 0}\left(\frac{a-1}{x}\right) = \log a \end{array}\right]$$

$$\lim_{x\to \frac{\pi}{2}} f(x) = \log 5 - 1$$

$f(\pi/2)$ is defined and $\lim_{x\to \frac{\pi}{2}} f(x)$ exists

But $\lim_{x\to \frac{\pi}{2}} f(x) \neq f\left(\frac{\pi}{2}\right)$

$\therefore$ the function $f(x)$ has removable discontinuity. This discontinuity can be removed by redefining $f(\pi/2) = \log 5 - 1$.

So the function can be redefined as follows

$$f(x) = \left(\frac{5^{\cos x} - e^{\left(\frac{\pi}{2}-x\right)}}{\cot x}\right), \text{ for } x \neq \frac{\pi}{2}$$

$$= \log 5 - 1, \text{ for } x = \frac{\pi}{2}$$

Ex. 11 : If $f(x) = \left(\frac{3x+2}{2-5x}\right)^{\frac{1}{x}}$, for $x \neq 0$,

is continuous at $x = 0$ then find $f(0)$

Solution : Given that $f(x)$ is continuous at $x = 0$

$$\therefore f(0) = \lim_{x\to 0}\left(\frac{3x+2}{2-5x}\right)^{\frac{1}{x}}$$

$$= \lim_{x\to 0}\left(\frac{2\left(1+\frac{3x}{2}\right)}{2\left(1-\frac{5x}{2}\right)}\right)^{\frac{1}{x}}$$

$$= \lim_{x\to 0}\left(\frac{\left(1+\frac{3x}{2}\right)^{\frac{1}{x}}}{\left(1-\frac{5x}{2}\right)^{\frac{1}{x}}}\right)$$

$$= \frac{\left[\lim_{x\to 0}\left(1+\frac{3x}{2}\right)^{\frac{2}{3x}}\right]^{\frac{3}{2}}}{\left[\lim_{x\to 0}\left(1-\frac{5x}{2}\right)^{\frac{-2}{5x}}\right]^{\frac{-5}{2}}}$$

$$= \frac{e^{\frac{3}{2}}}{e^{\frac{-5}{2}}} \qquad \left[\lim_{x\to 0}(1+kx)^{\frac{1}{kx}} = e\right]$$

$f(0) = e^4$

Ex. 12 : If $f(x)$ is defined on R, discuss the continuity of f at $x = \frac{\pi}{2}$, where

$$f(x) = \frac{5^{\cos x} + 5^{-\cos x} - 2}{(3\cot x).\log\left(\frac{2+\pi-2x}{2}\right)}, \text{ for } x \neq \frac{\pi}{2}$$

$$= \frac{2\log 5}{3}, \text{ for } x = \frac{\pi}{2}.$$

Solution : Given that, for $x = \frac{\pi}{2}$,

$$f\left(\frac{\pi}{2}\right) = \frac{2\log 5}{3},$$

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}}\left[\frac{5^{\cos x} + 5^{-\cos x} - 2}{3(\cot x)\log\left(\frac{2+\pi-2x}{2}\right)}\right]$$

$$= \lim_{x\to\frac{\pi}{2}}\left[\frac{5^{\cos x} + 5^{-\cos x} - 2}{3(\cot x)\log\left(1+\frac{\pi-2x}{2}\right)}\right]$$

Let $\frac{\pi-2x}{2} = t \Rightarrow x = \frac{\pi}{2} - t$ as $x \to \frac{\pi}{2}$, $t \to 0$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)$$

$$= \lim_{t\to 0}\left(\frac{5^{\cos\left(\frac{\pi}{2}-t\right)} + 5^{-\cos\left(\frac{\pi}{2}-t\right)} - 2}{3\cot\left(\frac{\pi}{2}-t\right).\log(1+t)}\right)$$

$$= \lim_{t\to 0}\left(\frac{5^{\sin t} + 5^{-\sin t} - 2}{3(\tan t).\log(1+t)}\right)$$

$$= \lim_{t\to 0}\left(\frac{5^{\sin t}\left(5^{\sin t} + 5^{-\sin t} - 2\right)}{3(\tan t)\log(1+t).5^{\sin t}}\right)$$

.........[Multiply and divide by $5^{\sin t}$]

$$= \lim_{t\to 0}\left(\frac{(5^{\sin t})^2+1-2\times 5^{\sin t}}{3(\tan t)\log(1+t)(5^{\sin t})}\right)$$

$$= \lim_{t\to 0}\left(\frac{(5^{\sin t}-1)^2}{3(\tan t).\log(1+t).(5^{\sin t})}\right)$$

$$= \lim_{t\to 0}\left[\frac{\frac{(5^{\sin t}-1)^2}{\sin^2 t}\times \sin^2 t}{3(\tan t).\log(1+t).(5^{\sin t})}\right]$$

t is small but $t \neq 0$. Hence $\sin t \neq 0$

So we can multiply and divide the numerator by $\sin^2 t$

$$= \lim_{t\to 0}\left(\frac{\left(\frac{5^{\sin t}-1}{\sin t}\right)^2\times\left(\frac{\sin t}{t}\right)^2}{3\left(\frac{\tan t}{t}\right).\frac{\log(1+t)}{t}.5^{\sin t}}\right)$$

[Dividing Numerator and Denominator by t^2 as $t \neq 0$]

$$= \frac{\left[\lim_{t\to 0}\left(\frac{5^{\sin t}-1}{\sin t}\right)\right]^2\times\lim_{t\to 0}\left(\frac{\sin t}{t}\right)^2}{3\lim_{t\to 0}\left(\frac{\tan t}{t}\right).\lim_{t\to 0}\left(\frac{\log(1+t)}{t}\right)}\times\lim_{t\to 0}\left(\frac{1}{5^{\sin t}}\right)$$

$$= \frac{(\log 5)^2\times(1)}{3(1)(1)}\times\frac{1}{5^0}\quad\left[\begin{array}{l}\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1, \lim_{\theta\to 0}\left(\frac{\tan\theta}{\theta}\right)=1\\ \lim_{x\to 0}\left(\frac{\log(1+x)}{x}\right)=1, \lim_{x\to 0}\left(\frac{a^x-1}{x}\right)=\log a\end{array}\right]$$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x) = \frac{(\log 5)^2}{3}$$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x) \neq f\left(\frac{\pi}{2}\right)$$

$\therefore$ $f(x)$ is discontinuous at $x = \left(\frac{\pi}{2}\right)$

Activity 1 :

Discuss the continuity of $f(x)$

where $f(x) = \dfrac{\log x-\log 5}{x-5}$ for $x \neq 5$

$= \dfrac{1}{5}$ for $x = 5$

Solution. : Given that $f(5) = \boxed{}$ .... (I)

$$\therefore \lim_{x\to 5} f(x) = \lim_{x\to 5}\left[\frac{\log x-\log 5}{x-5}\right]$$

put $x-5=t$ $\therefore$ $x = 5+t$. As $x \to 5$, $t \to 0$

$$= \lim_{t\to\square}\left[\frac{\log(\square)-\log 5}{t}\right]$$

$$= \lim_{t\to\square}\left[\frac{\log\left[\frac{\square}{5}\right]^t}{t}\right]$$

$$= \lim_{t\to\square}\left[\frac{\log\left(1+\frac{t}{5}\right)}{t}\right]$$

$$= \lim_{t\to\square}\left[\frac{\log\left(1+\frac{t}{5}\right)}{\frac{t}{\square}}\right]\times\frac{1}{\square}$$

$$= 1\times\frac{1}{\square}\quad\left[\lim_{x\to 0}\left(\frac{\log(1+px)}{px}\right)=1\right]$$

$$\therefore \lim_{x\to 0} f(x) = \frac{1}{\square}\quad\text{........... (II)}$$

$\therefore$ from (I) and (II)

$$\lim_{x\to 5} f(x) = f(5)$$

$\therefore$ The function $f(x)$ is continuous at $x = 5$.

8.1.11 THE INTERMEDIATE VALUE THEOREM FOR CONTINUOUS FUNCTIONS

Theorem : If f is a continuous function on a closed interval [a, b], and if y_0 is any value between f(a) and f(b) then $y_0 = f(c)$ for some c in [a, b].

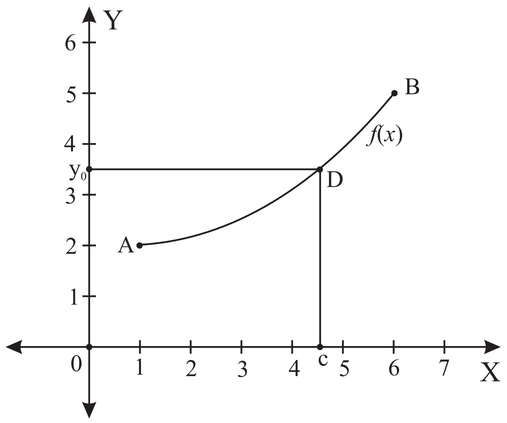

Fig. 8.10

Geometrically, the Intermediate Value Theorem says that any horizontal line $y = y_0$ crossing the Y-axis between the numbers f(a) and f(b) will cross the curve $y = f(x)$ at least once over the interval [a, b]. The proof of the Intermediate Value Theorem depends on the completeness property of the real number system and can be found in more advanced texts. The continuity of f on the interval is essential. If f is discontinuous at even one point of the interval, the conclusion of the theorem may fail.

Illustration 1 : Show that there is a root for the equation $x^3 - x - 1 = 0$ between 1 and 2

Solution : Let $f(x) = x^3 - x - 1$. $f(x)$ is a polynomial function so, it is continuous everywhere. We say that root of $f(x)$ exists if

$f(x) = 0$ for some value of x.

For $x = 1$, $f(1) = 1^3 - 1 - 1 = -1 < 0$

so, $f(1) < 0$

For $x = 2$, $f(2) = 2^3 - 2 - 1 = 5 > 0$ so, $f(2) > 0$

So by intermediate value theorem there has to be a point c between 1 and 2 with $f(c) = 0$.

Hence there is a root for the equation $x^3 - x - 1 = 0$ between 1 and 2.

EXERCISE 8.1

1) Examine the continuity of

(i) $f(x) = x^3 + 2x^2 - x - 2$ at $x = -2$.

(ii) $f(x) = \sin x$, for $x \le \frac{\pi}{4}$

$= \cos x$, for $x > \frac{\pi}{4}$, at $x = \frac{\pi}{4}$

(iii) $f(x) = \frac{x^2 - 9}{x - 3}$, for $x \ne 3$

$= 8$ for $x = 3$

2) Examine whether the function is continuous at the points indicated against them.

(i) $f(x) = x^3 - 2x + 1$, if $x \le 2$

$= 3x - 2$, if $x > 2$, at $x = 2$.

(ii) $f(x) = \frac{x^2 + 18x - 19}{x - 1}$, for $x \ne 1$

$= 20$ for $x = 1$, at $x = 1$

(iii) $f(x) = \frac{x}{\tan 3x} + 2$, for $x < 0$

$= \frac{7}{3}$, for $x \ge 0$, at $x = 0$.

3) Find all the points of discontinuities of $f(x) = \lfloor x \rfloor$ on the interval (–3, 2).

4) Discuss the continuity of the function $f(x) = |2x + 3|$, at $x = -3/2$

5) Test the continuity of the following functions at the points or interval indicated against them.

(i) $f(x) = \dfrac{\sqrt{x-1}-(x-1)^{\frac{1}{3}}}{x-2}$, for $x \neq 2$

$= \dfrac{1}{5}$, for $x = 2$

at $x = 2$

(ii) $f(x) = \dfrac{x^3-8}{\sqrt{x+2}-\sqrt{3x-2}}$ for $x \neq 2$

$= -24$ for $x = 2$ at $x = 2$

(iii) $f(x) = 4x + 1$, for $x \leq 8/3$.

$= \dfrac{59-9x}{3}$, for $x > 8/3$, at $x = 8/3$.

(iv) $f(x) = \dfrac{(27-2x)^{\frac{1}{3}}-3}{9-3(243+5x)^{\frac{1}{5}}}$, for $x \neq 0$

$= 2$ for $x = 0$, at $x = 0$

(v) $f(x) = \dfrac{x^2+8x-20}{2x^2-9x+10}$ for $0 < x < 3$; $x \neq 2$

$= 12$, for $x = 2$

$= \dfrac{2-2x-x^2}{x-4}$ for $3 \leq x < 4$

at $x = 2$

6) Identify discontinuities for the following functions as either a jump or a removable discontinuity.

(i) $f(x) = \dfrac{x^2-10x+21}{x-7}$.

(ii) $f(x) = x^2 + 3x - 2$, for $x \leq 4$

$= 5x + 3$, for $x > 4$.

(iii) $f(x) = x^2 - 3x - 2$, for $x < -3$

$= 3 + 8x$, for $x > -3$.

(iv) $f(x) = 4 + \sin x$, for $x < \pi$

$= 3 - \cos x$ for $x > \pi$

7) Show that following functions have continuous extension to the point where $f(x)$ is not defined. Also find the extension

(i) $f(x) = \dfrac{1-\cos 2x}{\sin x}$, for $x \neq 0$.

(ii) $f(x) = \dfrac{3\sin^2 x + 2\cos x(1-\cos 2x)}{2\left(1-\cos^2 x\right)}$, for $x \neq 0$.

(iii) $f(x) = \dfrac{x^2-1}{x^3+1}$ for $x \neq -1$.

8) Discuss the continuity of the following functions at the points indicated against them.

(i) $f(x) = \dfrac{\sqrt{3}-\tan x}{\pi-3x}$, $x \neq \dfrac{\pi}{3}$

$= \dfrac{3}{4}$, for $x = \dfrac{\pi}{3}$, at $x = \dfrac{\pi}{3}$.

(ii) $f(x) = \dfrac{e^{1/x}-1}{e^{1/x}+1}$, for $x \neq 0$

$= 1$, for $x = 0$, at $x = 0$.

(iii) $f(x) = \dfrac{4^x-2^{x+1}+1}{1-\cos 2x}$, for $x \neq 0$

$= \dfrac{(\log 2)^2}{2}$, for $x = 0$, at $x = 0$.

9) Which of the following functions has a removable discontinuity? If it has a removable discontinuity, redefine the function so that it becomes continuous.

(i) $f(x) = \dfrac{e^{5\sin x}-e^{2x}}{5\tan x-3x}$, for $x \neq 0$

$= 3/4$, for $x = 0$, at $x = 0$.

(ii) $f(x) = \log_{(1+3x)}(1+5x)$ for $x > 0$

$= \dfrac{32^x - 1}{8^x - 1}$, for $x < 0$, at $x = 0$.

(iii) $f(x) = \left(\dfrac{3-8x}{3-2x}\right)^{\frac{1}{x}}$, for $x \neq 0$.

(iv) $f(x) = 3x + 2$, for $-4 \leq x \leq -2$

$= 2x - 3$, for $-2 < x \leq 6$.

(v) $f(x) = \dfrac{x^3 - 8}{x^2 - 4}$, for $x > 2$

$= 3$, for $x = 2$

$= \dfrac{e^{3(x-2)^2} - 1}{2(x-2)^2}$, for $x < 2$

10) (i) If $f(x) = \dfrac{\sqrt{2+\sin x} - \sqrt{3}}{\cos^2 x}$, for $x \neq \dfrac{\pi}{2}$,

is continuous at $x = \dfrac{\pi}{2}$ then find $f\left(\dfrac{\pi}{2}\right)$.

(ii) If $f(x) = \dfrac{\cos^2 x - \sin^2 x - 1}{\sqrt{3x^2+1} - 1}$ for $x \neq 0$,

is continuous at $x = 0$ then find $f(0)$.

(iii) If $f(x) = \dfrac{4^{x-\pi} + 4^{\pi - x} - 2}{(x-\pi)^2}$ for $x \neq \pi$,

is continuous at $x = \pi$, then find $f(\pi)$.

11) (i) If $f(x) = \dfrac{24^x - 8^x - 3^x + 1}{12^x - 4^x - 3^x + 1}$, for $x \neq 0$

$= k$, for $x = 0$

is continuous at $x = 0$, find k.

(ii) If $f(x) = \dfrac{5^x + 5^{-x} - 2}{x^2}$, for $x \neq 0$

$= k$ for $x = 0$

is continuous at $x = 0$, find k.

(iii) If $f(x) = \dfrac{\sin 2x}{5x} - a$, for $x > 0$

$= 4$ for $x = 0$

$= x^2 + b - 3$, for $x < 0$

is continuous at $x = 0$, find a and b.

(iv) For what values of a and b is the function

$f(x) = ax + 2b + 18$, for $x \leq 0$

$= x^2 + 3a - b$, for $0 < x \leq 2$

$= 8x - 2$, for $x > 2$,

continuous for every x ?

(v) For what values of a and b is the function

$f(x) = \dfrac{x^2 - 4}{x - 2}$, for $x < 2$

$= ax^2 - bx + 3$, for $2 \leq x < 3$

$= 2x - a + b$, for $x \geq 3$

continuous for every x on R?

12) Discuss the continuity of f on its domain, where

$f(x) = |x + 1|$, for $-3 \leq x \leq 2$

$= |x - 5|$, for $2 < x \leq 7$.

13) Discuss the continuity of $f(x)$ at $x = \dfrac{\pi}{4}$ where,

$f(x) = \dfrac{(\sin x + \cos x)^3 - 2\sqrt{2}}{\sin 2x - 1}$, for $x \neq \dfrac{\pi}{4}$

$= \dfrac{3}{\sqrt{2}}$, for $x = \dfrac{\pi}{4}$.

14) Determine the values of p and q such that the following function is continuous on the entire real number line.

$f(x) = x + 1$, for $1 < x < 3$

$= x^2 + px + q$, for $|x - 2| \geq 1$.

15) Show that there is a root for the equation $2x^3 - x - 16 = 0$ between 2 and 3.

16) Show that there is a root for the equation $x^3 - 3x = 0$ between 1and 2.

17) Activity : Let $f(x) = ax + b$ (where a and b are unknown)

$$= x^2 + 5 \quad \text{for } x \in R$$

Find the values of a and b, so that $f(x)$ is continuous at $x = 1$. (Fig. 8.11)

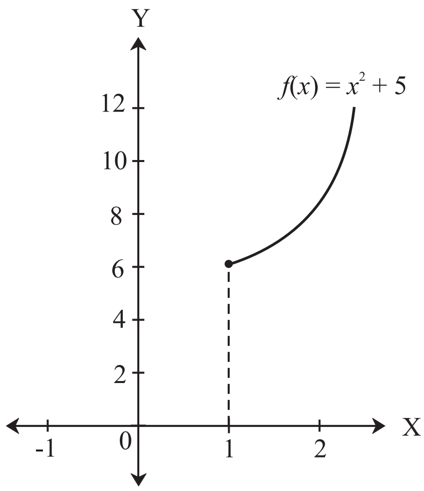

Fig. 8.11

18) Activity : Suppose $f(x) = px + 3$ for $a \le x \le b$

$$= 5x^2 - q \quad \text{for } b < x \le c$$

Find the condition on p, q, so that $f(x)$ is continuous on $[a,c]$, by filling in the boxes.

$f(b) = \boxed{}$

$\lim\limits_{x \to b^+} f(x) = \boxed{}$

$\therefore pb + 3 = \boxed{} - q$

$\therefore p = \dfrac{\boxed{}}{b}$ is the required condition.

Let's Remember

Continuity at a point

A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied:

(1) $f(a)$ is defined, (2) $\lim\limits_{x \to a} f(x)$ exists, and

(3) $\lim\limits_{x \to a} f(x) = f(a)$

Continuity from right : A function is continuous from right at a if $\lim\limits_{x \to a^+} f(x) = f(a)$

Continuity from left : A function is continuous from left at b if $\lim\limits_{x \to b^-} f(x) = f(b)$

Continuity over an interval :

Open Interval : A function is continuous over an open interval if it is continuous at every point in the interval.

Closed Interval : A function $f(x)$ is continuous over a closed interval $[a,b]$ if it is continuous at every point in (a,b),and it is continuous from right at a and from left at b.

Discontinuity at a point :

A function is discontinuous at a point or has a pont of discontinuity if it is not continuous at that point

Infinite discontinuity :

An infinite discontinuity occurs at a point a if

$$\lim_{x \to a-} f(x) = \pm\infty \text{ or } \lim_{x \to a^+} f(x) = \pm\infty$$

Jump discontinuity :

A jump discontinuity occurs at a point 'a' if

$\lim\limits_{x \to a-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ both exist, but

$$\lim_{x \to a-} f(x) \ne \lim_{x \to a^+} f(x)$$

Removable discontinuity :

A removable discontinuity occurs at a point a if $f(x)$ is discontinuous at a, but $\lim_{x \to a} f(x)$ exists and $f(a)$ may or may not be defined.

Intermediate Value Theorem :

Let f be continuous over a closed bounded interval [a,b]. If z is any real number between

$f(a)$ and $f(b)$, then there is a number c in $[a,b]$ satisfying $f(c) = z$.

MISCELLANEOUS EXERCISE-8

(I) Select the correct answer from the given alternatives.

(1) $f(x) = \dfrac{2^{\cot x} - 1}{\pi - 2x}$, for $x \neq \dfrac{\pi}{2}$

$= \log \sqrt{2}$, for $x = \dfrac{\pi}{2}$

(A) f is continuous at $x = \dfrac{\pi}{2}$

(B) f has a jump discontinuity at $x = \dfrac{\pi}{2}$

(C) f has a removable discontinuity

(D) $\lim_{x \to \frac{\pi}{2}} f(x) = 2\log 3$

(2) If $f(x) = \dfrac{1 - \sqrt{2}\sin x}{\pi - 4x}$, for $x \neq \dfrac{\pi}{4}$

is continuous at $x = \dfrac{\pi}{4}$, then $f\left(\dfrac{\pi}{4}\right) =$

(A) $\dfrac{1}{\sqrt{2}}$ (B) $-\dfrac{1}{\sqrt{2}}$ (C) $-\dfrac{1}{4}$ (D) $\dfrac{1}{4}$

(3) If $f(x) = \dfrac{(\sin 2x)\tan 5x}{(e^{2x} - 1)^2}$, for $x \neq 0$

is continuous at $x = 0$, then $f(0)$ is

(A) $\dfrac{10}{e^2}$ (B) $\dfrac{10}{e^4}$ (C) $\dfrac{5}{4}$ (D) $\dfrac{5}{2}$

(4) $f(x) = \dfrac{x^2 - 7x + 10}{x^2 + 2x - 8}$, for $x \in [-6, -3]$

(A) f is discontinuous at $x = 2$.

(B) f is discontinuous at $x = -4$.

(C) f is discontinuous at $x = 0$.

(D) f is discontinuous at $x = 2$ and $x = -4$.

(5) If $f(x) = ax^2 + bx + 1$, for $|x - 1| \geq 3$ and

$= 4x + 5$, for $-2 < x < 4$

is continuous everywhere then,

(A) $a = \dfrac{1}{2}$, $b = 3$ (B) $a = -\dfrac{1}{2}$, $b = -3$

(C) $a = -\dfrac{1}{2}$, $b = 3$ (D) $a = \dfrac{1}{2}$, $b = -3$

(6) $f(x) = \dfrac{(16^x - 1)(9^x - 1)}{(27^x - 1)(32^x - 1)}$, for $x \neq 0$

$= k$, for $x = 0$

is continuous at $x = 0$, then 'k' $=$

(A) $\dfrac{8}{3}$ (B) $\dfrac{8}{15}$ (C) $-\dfrac{8}{15}$ (D) $\dfrac{20}{3}$

(7) $f(x) = \dfrac{32^x - 8^x - 4^x + 1}{4^x - 2^{x+1} + 1}$, for $x \neq 0$

$= k$, for $x = 0$,

is continuous at $x = 0$, then value of 'k' is

(A) 6 (B) 4 (C) (log2)(log4) (D) 3log4

(8) If $f(x) = \dfrac{12^x - 4^x - 3^x + 1}{1 - \cos 2x}$, for $x \neq 0$ is continuous at $x = 0$ then the value of $f(0)$ is

(A) $\dfrac{\log 12}{2}$ (B) log2.log3

(C) $\dfrac{\log 2 . \log 3}{2}$ (D) None of these.

(9) If $f(x) = \left(\dfrac{4+5x}{4-7x}\right)^{\frac{4}{x}}$, for $x \neq 0$ and $f(0) = k$, is continuous at $x = 0$, then k is

(A) e^7 (B) e^3 (C) e^{12} (D) $e^{\frac{3}{4}}$

(10) If $f(x) = \lfloor x \rfloor$ for $x \in (-1,2)$ then f is discontinuous at

(A) $x = -1, 0, 1, 2,$ (B) $x = -1, 0, 1$

(C) $x = 0, 1$ (D) $x = 2$

(II) Discuss the continuity of the following functions at the point(s) or on the interval indicated against them.

(1) $f(x) = \dfrac{x^2-3x-10}{x-5}$, for $3 \le x \le 6, x \neq 5$

$= 10$, for $x = 5$

$= \dfrac{x^2-3x-10}{x-5}$, for $6 < x \le 9$

(2) $f(x) = 2x^2 - 2x + 5$, for $0 \le x \le 2$

$= \dfrac{1-3x-x^2}{1-x}$, for $2 < x < 4$

$= \dfrac{x^2-25}{x-5}$, for $4 \le x \le 7$ and $x \neq 5$

$= 7$ for $x = 5$

(3) $f(x) = \dfrac{\cos 4x - \cos 9x}{1-\cos x}$, for $x \neq 0$

$f(0) = \dfrac{68}{15}$, at $x = 0$ on $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

(4) $f(x) = \dfrac{\sin^2 \pi x}{3(1-x)^2}$, for $x \neq 1$

$= \dfrac{\pi^2 \sin^2\left(\frac{\pi x}{2}\right)}{3+4\cos^2\left(\frac{\pi x}{2}\right)}$ for $x = 1$, at $x = 1$.

(5) $f(x) = \dfrac{|x+1|}{2x^2+x-1}$, for $x \neq -1$

$= 0$ for $x = -1$ at $x = -1$.

(6) $f(x) = [x+1]$ for $x \in [-2, 2)$

Where [*] is greatest integer function.

(7) $f(x) = 2x^2 + x + 1$, for $|x-3| \ge 2$

$= x^2+3$, for $1 < x < 5$

(III) Identify discontinuities if any for the following functions as either a jump or a removable discontinuity on their respective domains.

(1) $f(x) = x^2 + x - 3$, for $x \in [-5, -2)$

$= x^2 - 5$, for $x \in (-2, 5]$

(2) $f(x) = x^2 + 5x + 1$, for $0 \le x \le 3$

$= x^3 + x + 5$, for $3 < x \le 6$

(3) $f(x) = \dfrac{x^2+x+1}{x+1}$, for $x \in [0, 3)$

$= \dfrac{3x+4}{x^2-5}$, for $x \in [3, 6]$.

(IV) Discuss the continuity of the following functions at the point or on the interval indicated against them. If the function is discontinuous, identify the type of discontinuity and state whether the discontinuity is removable. If it has a removable discontinuity, redefine the function so that it becomes continuous.

(1) $f(x) = \dfrac{(x+3)(x^2-6x+8)}{x^2-x-12}$

(2) $f(x) = x^2 + 2x + 5$, for $x \le 3$

$= x^3 - 2x^2 - 5$, for $x > 3$

(V) Find k if following functions are continuous at the points indicated against them.

(1) $f(x) = \left(\frac{5x-8}{8-3x}\right)^{\frac{3}{2x-4}}$, for $x \neq 2$

$= k$, for $x = 2$ at $x = 2$.

(2) $f(x) = \frac{45^x - 9^x - 5^x + 1}{(k^x - 1)(3^x - 1)}$, for $x \neq 0$

$= \frac{2}{3}$, for $x = 0$, at $x = 0$

(VI) Find a and b if following functions are continuous at the points or on the interval indicated against them.

(1) $f(x) = \frac{4\tan x + 5\sin x}{a^x - 1}$,for $x < 0$

$= \frac{9}{\log 2}$, for $x = 0$

$= \frac{11x + 7x.\cos x}{b^x - 1}$, for $x > 0$.

(2) $f(x) = ax^2 + bx + 1$, for $|2x - 3| \geq 2$

$= 3x + 2$, for $\frac{1}{2} < x < \frac{5}{2}$.

(VII) Find $f(a)$, if f is continuous at $x = a$ where,

(1) $f(x) = \frac{1 + \cos(\pi x)}{\pi(1-x)^2}$, for $x \neq 1$ and at $a = 1$.

(2) $f(x) = \frac{1 - \cos[7(x-\pi)]}{5(x-\pi)^2}$, for $x \neq \pi$ at $a = \pi$.

(VIII) Solve using intermediate value theorem.

(1) Show that $5^x - 6x = 0$ has a root in $[1, 2]$

(2) Show that $x^3 - 5x^2 + 3x + 6 = 0$ has at least two real roots between $x = 1$ and $x = 5$.

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