9 Probability

Let's Learn

- Basic Terminologies
- Concept of probability
- Addition Theorem
- Conditional probability
- Multiplication Theorem
- Baye's Theorem
- Odds

Let's Recall

9.1.1 Basic Terminologies

Random Experiment : Suppose an experiment having more than one outcome. All possible results are known but the actual result cannot be predicted such an experiment is called a random experiment.

Outcome: A possible result of random experiment is called a possible outcome of the experiment.

Sample space: The set of all possible outcomes of a random experiment is called the sample space. The sample space is denoted by S or Greek letter omega (Ω). The number of elements in S is denoted by n(S). A possible outcome is also called a sample point since it is an element in the sample space.

Event: A subset of the sample space is called an event.

Favourable Outcome: An outcome that belongs to the specified event is called a favourable outcome.

Types of Events:

Elementary Event: An event consisting of a single outcome is called an elementary event.

Certain Event: The sample space is called the certain event if all possible outcomes are favourable outcomes. i.e. the event consists of the whole sample space.

Impossible Event: The empty set is called impossible event as no possible outcome is favorable.

Algebra of Events:

Events are subsets of the sample space. Algebra of events uses operations in set theory to define new events in terms of known events.

Union of Two Events: Let A and B be two events in the sample space S. The union of A and B is denoted by $A \cup B$ and is the set of all possible outcomes that belong to at least one of A and B.

Ex. Let S = Set of all positive integers not exceeding 50;

Event A = Set of elements of S that are divisible by 6; and

Event B = Set of elements of S that are divisible by 9. Find $A \cup B$

Solution : $A = \{6,12,18,24,30,36,42,48\}$

$B = \{9,18,27,36,45\}$

$\therefore\ A \cup B = \{6,9,12,18,24,27,30,36,42,45, 48\}$ is the set of elements of S that are divisible by 6 or 9.

Exhaustive Events: Two events A and B in the sample space S are said to be exhaustive if $A \cup B = S$.

Example: Consider the experiment of throwing a die and noting the number on the top.

Let S be the sample space

$\therefore$ S = {1,2,3,4,5,6}

Let, A be the event that this number does not exceed 4, and

B be the event that this number is not smaller than 3.

Then A = {1,2,3,4} B = {3,4,5,6}

and therefore, $A \cup B$ = {1,2,3,4,5,6} = S

$\therefore$ Events A and B are exhaustive.

Intersection of Two Events: Let A and B be two events in the sample space S. The intersection of A and B is the event consisting of outcomes that belong to both the events A and B.

Example, Let S = Set of all positive integers not exceeding 50,

Event A = Set of elements of S that are divisible by 3, and

Event B = Set of elements of S that are divisible by 5.

Then A = {3,6,9,12,15,18,21,24,27,30,33, 36,39,42,45,48},

B = {5,10,15,20,25,30,35,40,45,50}

$\therefore$ $A \cap B$ = {15,30,45} is the set of elements of S that are divisible by both 3 and 5.

Mutually Exclusive Events: Event A and B in the sample space S are said to be mutually exclusive if they have no outcomes in common. In other words, the intersection of mutually exclusive events is empty. Mutually exclusive events are also called disjoint events.

Example: Let S = Set of all positive integers not exceeding 50,

Event A = set of elements of S that are divisible by 8, and

Event B = set of elements of S that are divisible by 13.

Then A = {8,16,24,32,40,48},

B = {13,26,39}

$\therefore$ $A \cap B = \phi$ because no element of S is divisible by both 8 and 13.

Note: If two events A and B are mutually exclusive and exhaustive, then they are called complementary events.

Symbolically, A and B are complementary events if $A \cup B$ = S and $A \cap B = \phi$.

Notation: Complement of an event A is denoted by A', $\bar{A}$ or A^c. The following table shows how the operations of complement, union, and intersection can be combined to define more events.

Operation	Interpretation
A', $\bar{A}$ or A^c	Not A.
$A \cup B$	At least, one of A and B
$A \cap B$	Both A and B
$(A' \cap B) \cup (A \cap B')$	Exactly one of A and B
$(A' \cap B') = (A \cup B)'$	Neither A nor B

SOLVED EXAMPLES:

Ex. 1: Describe the sample space of the experiment when a coin and a die are thrown simultaneously.

Solution : Sample space S = {(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)}

Ex. 2: Sunita and Samrudhi who live in Mumbai wish to go on holiday to Delhi together. They can travel to Delhi from Mumbai either by car or by train or plane and on reaching Delhi they can go for city-tour either by bus or Taxi. Describe the sample space, showing all the combined outcomes of different ways they could complete city-tour from Mumbai.

Solution : Sample space

S = {(car, bus), (car, taxi), (train, bus), (train, taxi), (plane, bus), (plane, taxi)}

Ex. 3: Three coins are tossed. Events E_1, E_2, E_3 and E_4 are defined as follows.

E_1 : Occurence of at least two heads.

E_2 : Occurence of at least two tails.

E_3 : Occurence of at most one head.

E_4 : Occurence of two heads.

Describe the sample space and events E_1, E_2, E_3 and E_4.

Find $E_1 \cup E_4$, E_3'. Also check whether

i) E_1 and E_2 are mutually exclusive

ii) E_2 and E_3 are equal

Solution : Sample space

S= {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

E_1: {HHH, HHT, HTH, THH}

E_2: {HTT, THT, TTH, TTT}

E_3: {HTT, THT, TTH, TTT}

E_4: {HHT, HTH, THH,}

$E_1 \cup E_4$ = {HTT, HTH, THH, HHH, HHT}

E_3 = {HHH, HHT, HTH, THH}

i) $E_1 \cap E_2 = \{\} = \phi$

∴ E_1 and E_2 are mutually exclusive.

ii) E_2 and E_3 are equal.

9.1.2 Concept of Probability:

A random experiment poses uncertainty regarding the actual result of the experiment, even though all possible outcomes are already known. The classical definition of probability is based on the assumption that all possible outcomes of an experiment are equally likely.

9.1.3 Equally likely outcomes:

All possible outcomes of a random experiment are said to be equally likely if none of them can be preferred over others.

9.1.4 Probability of an Event:

The probability of an event A is defined as

$$P(A) = \frac{n(A)}{n(S)}$$

Where,

n(A) = number of outcomes favorable for event A,

n(S) = number of all possible outcomes.

9.1.5 Elementary Properties of Probabilty:

1) A' is complement of A and therefore $P(A') = 1 - P(A)$

2) For any event A in S, $0 \le P(A) \le 1$

3) For the impossible event ϕ, $P(\phi) = 0$

4) For the certain event S, $P(S) = 1$

5) If A_1 and A_2 two mutually exclusive events then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

6) If $A \subseteq B$, then $P(A) \le P(B)$ and $P(A' \cap B) = P(B) - P(A)$

7) Addition theorem: For any two events A and B of a sample space S,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

8) For any two events A and B,

$$P(A \cap B') = P(A) - P(A \cap B)$$

9) For any three events A, B and C of a sample space S,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - (P(A \cap C) + P(A \cap B \cap C)$$

10) If A_1, A_2, ..., A_m are mutually exclusive events in S, then $P(A_1 \cup A_2 \cup, ... \cup A_m$ $= P(A_1) + P(A_2) + ... + P(A_m)$

Remark: Consider a finite sample space S with n finite elements.

$S = \{a_1, a_2, a_3, \ldots a_n\}$. Let $A_1, A_2, A_3, \ldots A_n$ be elementary events given by $A_i = \{a_i\}$ with probability $P(A_i)$. We have

$P(S) = P(A_1) + P(A_2) + \ldots + P(A_n) = 1 \quad \ldots(I)$

When all elementary events given by A_i (i = 1, 2, 3, ... n) are equally likely, that is $P(A_1) = P(A_2) = \ldots = P(A_n)$, then from (I), we have $P(A_i) = 1/n$, i = 1, 2,..... n

If A is any event made up of m such elementary events, i.e.

$A = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_m$, then using property 10, we have

$P(A) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_m)$

$$= \left(\frac{1}{n}\right) + \left(\frac{1}{n}\right) + \ldots + \left(\frac{1}{n}\right) \text{ (m times)}$$

$$\therefore P(A) = \frac{m}{n} = \frac{n(A)}{n(S)} \quad \ldots \text{(II)}$$

$\therefore$ P(A) = (Number of favourable outcomes for the occurrence of event A)/ (Total number of distinct possible outcomes in the sample space S)

SOLVED EXAMPLES

Ex. 1) If $A \cup B \cup C = S$ (the sample space) and A, B and C are mutually exclusive events, can the following represent probability assignment?

i) $P(A) = 0.2, P(B) = 0.7, P(C) = 0.1$

ii) $P(A) = 0.4, P(B) = 0.6, P(C) = 0.2$

Solution:

i) Since $P(S) = P(A \cup B \cup C)$

$= P(A) + P(B) + P(C)$ [Property 10]

$= 0.2 + 0.7 + 0.1 = 1$

and $0 \leq P(A), P(B), P(C) \leq 1$

$\therefore$ The given values can represent the probability assignment.

ii) Since

$P(S) = P(A \cup B \cup C)$

$= P(A) + P(B) + P(C)$ [Property 10]

$= 0.4 + 0.6 + 0.2 = 1.2 > 1$

$\therefore$ P(A), P(B) and P(C) cannot represent probability assignment.

Ex. 2) One card is drawn at random from a pack of 52 cards. What is the probability that it is a King or Queen?

Solution:

Random Experiment = One card is drawn at random from a pack of 52 cards

$\therefore n(S) = {}^{52}C_1 = 52.$

Let event A: Card drawn is King

and event B: Card drawn is Queen.

Since pack of 52 cards contains, 4 king cards from which any one king card can be drawn in ${}^4C_1 = 4$ ways. $\therefore n(A) = 4$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{4}{52}$$

Similarly, a pack of 52 cards contains, 4 queen cards from which any one queen card can be drawn in ${}^4C_1 = 4$ ways. $\therefore n(B) = 4$

$$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{4}{52}$$

Since A and B are mutually exclusive events

$\therefore$ required probability P(king or queen)

$$= P(A \cup B) = P(A) + P(B) = \frac{4}{52} + \frac{4}{52} = \frac{2}{13}$$

Ex. 3: Five employees in a company of 20 are graduates. If 3 are selected out of 20 at random. What is the probability that

i) they are all graduates?

ii) there is at least one graduate among them?

Solution : Out of 20 employees, any 3 are to be selected in ${}^{20}C_3$ ways.

$\therefore$ $n(S) = {}^{20}C_3$ where S is the sample space.

Let event A: All 3 selected employees are graduates.

Out of 5 graduate any 3 can be selected in ${}^{5}C_3$ ways.

$\therefore$ required probability $P(A) = \frac{{}^{5}C_3}{{}^{20}C_3} = \frac{10}{1140}$

$= \frac{1}{114}$

Let event B: At least one graduate employee is selected.

$\therefore$ B' is the event that no graduate employee is selected.

Since out of 20 employee, 5 are graduates, therefore from the remaining 15 non-graduate any 3 non-graduates can be selected in ${}^{15}C_3$ ways.

$$\therefore P(B') = \frac{{}^{15}C_3}{{}^{20}C_3} = \frac{455}{1140} = \frac{91}{228}$$

$\therefore$ required probability

$$P(B) = 1 - P(B') = 1 - \frac{91}{228} = \frac{137}{228}$$

Ex. 4) The letters of the word STORY be arranged randomly. Find the probability that

a) T and Y are together.

b) arrangment begins with T and end with Y.

Solution:

The word STORY consists of 5 different letters, which can be arranged among themselves in 5! ways.

$\therefore$ $n(S) = 5! = 120$

a) Event A: T and Y are a together.

Let us consider T and Y as a single letter say X. Therefore, now we have four different letters X, S, O and R which can be arranged among themselves in 4! = 24 different ways. After this is done, two letters T and Y can be arranged among themselves in 2! = 2 ways. Therefore, by fundamental principle, total number of arrangements in which T and Y are always together is $24 \times 2 = 48$.

$\therefore$ required probability $P(A) = \frac{48}{120} = \frac{2}{5}$

b) Event B: An arrangement begins with T and ends with Y.

Remaining 3 letters in the middle can be arranged in 3! = 6 different ways.

$\therefore$ required probability $P(B) = \frac{6}{120} = \frac{1}{120}$

EXERCISE 9.1

1) There are four pens: Red, Green, Blue and Purple in a desk drawer of which two pens are selected at random one after the other with replacement. State the sample space and the following events.
   a) A : Selecting at least one red pen.
   b) B : Two pens of the same color are not selected.

2) A coin and a die are tossed simultaneously. Enumerate the sample space and the following events.
   a) A : Getting a Tail and an Odd number
   b) B : Getting a prime number
   c) C : Getting a head and a perfect square.

3) Find n(S) for each of the following random experiments.
   a) From an urn containing 5 gold and 3 silver coins, 3 coins are drawn at random
   b) 5 letters are to be placed into 5 envelopes such that no envelop is empty.
   c) 6 books of different subjects arranged on a shelf.
   d) 3 tickets are drawn from a box containing 20 lottery tickets.

4) Two fair dice are thrown. State the sample space and write the favorable outcomes for the following events.

   a) A : Sum of numbers on two dice is divisible by 3 or 4.
   b) B : Sum of numbers on two dice is 7.
   c) C : Odd number on the first die.
   d) D : Even number on the first die.
   e) Check whether events A and B are mutually exclusive and exhaustive.
   f) Check whether events C and D are mutually exclusive and exhaustive.

5) A bag contains four cards marked as 5, 6, 7 and 8. Find the sample space if two cards are drawn at random

   a) with replacement
   b) without replacement

6) A fair die is thrown two times. Find the probability that

   a) sum of the numbers on them is 5
   b) sum of the numbers on them is at least 8
   c) first throw gives a multiple of 2 and second throw gives a multiple of 3.
   d) product of numbers on them is 12.

7) Two cards are drawn from a pack of 52 cards. Find the probability that

   a) one is a face card and the other is an ace card
   b) one is club and the other is a diamond
   c) both are from the same suit.
   d) both are red cards
   e) one is a heart card and the other is a non heart card

8) Three cards are drawn from a pack of 52 cards. Find the chance that

   a) two are queen cards and one is an ace card
   b) at least one is a diamond card
   c) all are from the same suit
   d) they are a king, a queen and a jack

9) From a bag containing 10 red, 4 blue and 6 black balls, a ball is drawn at random. Find the probability of drawing

   a) a red ball.
   b) a blue or black ball.
   c) not a black ball.

10) A box contains 75 tickets numbered 1 to 75. A ticket is drawn at random from the box. Find the probability that,

   a) Number on the ticket is divisible by 6
   b) Number on the ticket is a perfect square
   c) Number on the ticket is prime
   d) Number on the ticket is divisible by 3 and 5

11) What is the chance that a leap year, selected at random, will contain 53 sundays?.

12) Find the probability of getting both red balls, when from a bag containing 5 red and 4 black balls, two balls are drawn, i) with replacement ii) without replacement

13) A room has three sockets for lamps. From a collection 10 bulbs of which 6 are defective. At night a person selects 3 bulbs, at random and puts them in sockets. What is the probability that i) room is still dark ii) the room is lit

14) Letters of the word MOTHER are arranged at random. Find the probability that in the arrangement

   a) vowels are always together
   b) vowels are never together
   c) O is at the begining and end with T
   d) starting with a vowel and end with a consonant

15) 4 letters are to be posted in 4 post boxes. If any number of letters can be posted in any of the 4 post boxes, what is the probability that each box contains only one letter?

16) 15 professors have been invited for a round table conference by Vice chancellor of a university. What is the probability that two particular professors occupy the seats on either side of the Vice Chancellor during the conference.

17) A bag contains 7 black and 4 red balls. If 3 balls are drawn at random find the probability that (i) all are black (ii) one is black and two are red.

9.2.1 Addition theorem for two events

For any two events A and B of a sample space S, P(A∪B) = P(A) + P(B) − P(A∩B). This is the property (7) that we had seen earlier. Since it is very important we give its proof. The other properties can also be proved in the same way.

This can be proved by two methods

a) Using the definition of probability.

b) Using Venn diagram.

We assume that all outcomes are equally likely and sample space S contains finite number of outcomes.

(a) Using the definition of probability:

If A and B are any two events, then event A∪B can be decomposed into two mutually exclusive events A∩B' and B

i.e. A∪B = (A∩B')∪B

∴ P(A∪B) = P[(A∩B')∪B]

= P(A∩B') + P(B) [By property 10]

= P(A) – P(A∩B) + P(B) [By property 8]

Hence P(A∪B) = P(A) + P(B) − P(A∩B)

(b) Using Venn diagram:

Let n(S) = n = Total no. of distinct possible outcomes in the sample space S.

n(A) = x, number of favourable outcomes for the occurrence of event A.

n(B) = y, number of favourable outcomes for the occurrence of event B.

n(A∩B) = z, the number of favourable outcomes for the occurrence of both event A and B.

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{x}{n},\ P(B) = \frac{n(B)}{n(S)} = \frac{y}{n}$$

$$P(A\cap B) = \frac{n(A\cap B)}{n(S)} = \frac{z}{n}$$

As all outcomes are equally likely.

From Venn diagram

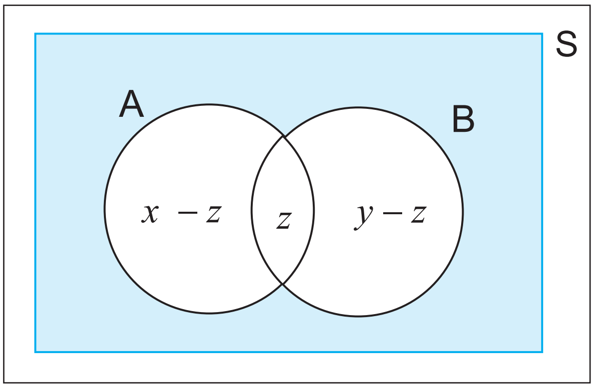


Fig. 9.1

n(A∪B) = $(x-z) + z + (y-z)$

∴ n(A∪B) = $x + y - z$

n(A∪B) = n(A) + n(B) – n(A∩B)

Dividing both sides by n(S), we get

$$\frac{n(A\cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A\cap B)}{n(S)}$$

∴ P(A∪B) = P(A) + P(B) – P(A∩B).

SOLVED EXAMPLES

Ex. 1) Two dice are thrown together. What is the probability that,

i) Sum of the numbers is divisible by 3 or 4?

ii) Sum of the numbers is neither divisible by 3 nor 5?

Solution : Let S be the sample space

Let $N_1 = N_2 = \{1,2,3,4,5,6\}$

$S = N_1 \times N_2 = \{(x,y)/x \in N_1, y \in N_2\}$

n(S) = 36

i) Let event A: Sum of the numbers is divisible by 3

$\therefore$ possible sums are 3, 6, 9, 12.

$\therefore$ A = {(1, 2), (1, 5), (2, 1), (2, 4), (3, 3), (3, 6), (4, 2), (4, 5), (5, 1), (5, 4), (6, 3), (6, 6)}

$\therefore$ n(A) = 12 $\therefore$ P(A) = n(A) / n(S) = 12/36

Let event B: Sum of the numbers is divisible by 4.

$\therefore$ possible sums are 4, 8, 12

$\therefore$ B ={(1, 3), (2, 2), (2, 6), (3, 1), (3, 5), (4, 4), (5, 3), (6, 2), (6, 6)}

$\therefore$ n(B) = 9 $\therefore$ P(B) = n(B) / n(S) = $\frac{9}{36}$

$\therefore$ Event $A \cap B$: Sum of the numbers is divisible by 3 and 4 i.e. divisible by 12.

$\therefore$ possible Sum is 12

$\therefore$ $A \cap B$ = {(6, 6)}

$\therefore$ $n(A \cap B) = 1$

$\therefore$ $P(A \cap B) = n(A \cap B)/n(S) = \frac{1}{36}$

P (Sum of the numbers is divisible by 3 or 4)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{12}{36} + \frac{9}{36} - \frac{1}{36} = \frac{20}{36} = \frac{5}{9}$

ii) Let event A: Sum of the numbers is divisible by 3

$\therefore P(A) = \frac{12}{36}$

Let event Y: Sum of the numbers is divisible by 5.

$\therefore$ possible sums are 5, 10

$\therefore$ Y ={(1, 4), (2, 3), (3, 2), (4, 1), (4, 6), (5, 5), (6, 4)}

$\therefore$ n(Y) = 7 $\therefore P(Y) = \frac{n(Y)}{n(S)} = \frac{7}{36}$

$\therefore$ Event $A \cap Y$: sum is divisible by 3 and 5

$\therefore$ $A \cap Y = \phi$

[X and Y are mutually exclusive events]

$\therefore P(A \cap Y) = \frac{n(A \cap Y)}{n(S)} = 0$

$\therefore$ required probability = P(Sum of the numbers is neither divisible by 3 nor 5)

$P(A' \cap Y') = P(A \cup Y)'$ [De'Morgan's law]

$= 1 - P(X \cup Y)$ [Property 1]

$= 1 - [P(A) + P(Y) - P(A \cap Y)]$

$= 1 - \frac{19}{36} = \frac{17}{36}$

Ex. 2) The probability that a student will solve problem A is 2/3, and the probability that he will not solve problem B is 5/9. If the probability that student solves at least one problem is 4/5, what is the probability that he will solve both the problems?

Solution : Let event A: student solves problem A

$\therefore P(A) = \frac{2}{3}$

event B: student solves problem B.

$\therefore$ event B': student will not solve problem B.

$\therefore P(B') = \frac{5}{9}$

$\therefore$ $P(B) = 1 - P(B') = 1 - \frac{5}{9} = \frac{4}{9}$

Probability that student solves at least one problem $= P(A \cup B) = \frac{4}{5}$

$\therefore$ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore$ required probability = P(he will solve both the problems)

$= P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= \frac{2}{3} + \frac{4}{9} - \frac{4}{5} = \frac{14}{45}$

EXERCISE 9.2

1) First 6 faced die which is numbered 1 through 6 is thrown then a 5 faced die which is numbered 1 through 5 is thrown. What is the probability that sum of the numbers on the upper faces of the dice is divisible by 2 or 3?

2) A card is drawn from a pack of 52 cards. What is the probability that,
   i) card is either red or black?
   ii) card is either black or a face card?

3) A girl is preparing for National Level Entrance exam and State Level Entrance exam for professional courses. The chances of her cracking National Level exam is 0.42 and that of State Level exam is 0.54. The probability that she clears both the exams is 0.11. Find the probability that (i) She cracks at least one of the two exams (ii) She cracks only one of the two (iii) She cracks none

4) A bag contains 75 tickets numbered from 1 to 75. One ticket is drawn at random. Find the probability that,
   a) number on the ticket is a perfect square or divisible by 4
   b) number on the ticket is a prime number or greater than 40

5) The probability that a student will pass in French is 0.64, will pass in Sociology is 0.45 and will pass in both is 0.40. What is the probability that the student will pass in at least one of the two subjects?

6) Two fair dice are thrown. Find the probability that number on the upper face of the first die is 3 or sum of the numbers on their upper faces is 6.

7) For two events A and B of a sample space S, if $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{2}$ and $P(A \cup B) = \frac{5}{8}$.

   Find the value of the following.
   a) $P(A \cap B)$
   b) $P(A' \cap B')$
   c) $P(A' \cup B')$

8) For two events A and B of a sample space S, if $P(A \cup B) = \frac{5}{6}$, $P(A \cap B) = \frac{1}{3}$ and $P(B') = \frac{1}{3}$, then find P(A).

9) A bag contains 5 red, 4 blue and an unknown number m of green balls. If the probability of getting both the balls green, when two balls are selected at random is $\frac{1}{7}$, find m.

10) Form a group of 4 men, 4 women and 3 children, 4 persons are selected at random. Find the probability that, i) no child is selected ii) exactly 2 men are selected.

11) A number is drawn at random from the numbers 1 to 50. Find the probability that it is divisible by 2 or 3 or 10.

9.3.1 Conditional Probability:

Let S be a sample space associated with the given random experiment. Let A and B be any two events defined on the sample space S. Then the probability of occurrence of event A under the condition that event B has already occurred and P(B) $\neq 0$ is called conditional probability of event A given B and is denoted by P(A/B).

SOLVED EXAMPLES

Ex.1) A card is drawn from a pack of 52 cards, given that it is a red card, what is the probability that it is a face card.

Solution : Let event A: Red card is drawn and event B: face card is drawn

A card is drawn from a pack of 52 cards, therefore n(S) = 52. But we are given that red card is drawn, therefore our sample space reduces to event A only, which contains n(A) = 26 sample points. Event A is called reduced or truncated sample space. Out of 26 red cards, 6 cards are favourable for face cards.

∴ P[card drawn is face card given that it is a red card] = P[B/A] = 6 /26 = 3/13

Ex. 2) A pair of dice is thrown. If sum of the numbers is an odd number, what is the probability that sum is divisible by 3?

Solution : Let Event A: sum is an odd number.

Event B: Sum is divisible by 3.

A pair of dice is thrown, therefore n(S) = 36. But we are given that sum is odd, therefore our sample space reduces to event A only as follows:

A = {(1,2), (1,4), (1,6), (2,1), (2,3), (2,5), (3,2), (3,4), (3,6), (4,1), (4,3), (4,5), (5,2), (5,4), (5,6), (6,1), (6,3), (6,5)}

∴ n(A) = 18

Out of 18 sample points following 6 sample points are favourable for occurrence of event B

B = {(1, 2), (2, 1), (3, 6), (4, 5), (5, 4), (6, 3)}

∴ P[sum is divisible by 3 given that sum is an odd number] = P(B/A) = 6/18 = 1/3

9.3.2 Let S be a finite sample space, associated with the given random experiment, containing equally likely outcomes. Then we have the following result.

Statement: Conditional probability of event A given that event B has already occurred is given by

$$P(A/B) = \frac{P(A \cap B)}{P(B)},\ P(B) \neq 0$$

(Read A/B as A given B)

Let S be a sample space associated with the given random experiment and n(S) be the number of sample points in the sample space S. Since we are given that event B has already occurred, therefore our sample space reduces to event B only, which contains n(B) sample points. Event B is also called reduced or truncated sample space. Now out of n(B) sample points, only $n(A \cap B)$ sample points are favourable for occurrence of event A. Therefore, by definition of probability

$$P(A/B) = \frac{n(A \cap B)}{n(B)},\ n(B) \neq 0$$

$$= \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}}$$

$$\therefore P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

$$\text{Similarly } P(B/A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) \neq 0$$

SOLVED EXAMPLES

Ex. 1: Find the probability that a single toss of a die will result in a number less than 4 if it is given that the toss resulted in an odd number.

Solution : Let event A: toss resulted in an odd number and

Event B: number is less than 4

∴ A = {1, 3, 5} ∴ P(A) = 3/6 = $\frac{1}{2}$

B = {1, 2, 3} ∴ A∩B = {1, 3}

∴ $P(A\cap B) = \frac{2}{6} = \frac{1}{3}$

∴ P(number is less than4 given that it is odd)

$= P(B/A) = P(A\cap B)/\ P(A) = \left(\frac{1}{3}\right)/\left(\frac{1}{2}\right) = \frac{2}{3}$

Ex. 2) If P(A') = 0.7, P(B) = 0.7, P(B/A) = 0.5, find P(A/B) and P(A∪B).

Solution : Since 1 – P(A') = 0.7

P(A) = 1 – P(A') = 1 − 0.7 = 0.3

Now P(B/A) = P(A∩B)/P(A)

∴ 0.5 = P(A∩B)/ 0.3

∴ P(A∩B) = 0.15

Again P(A/B) = P(A∩B)/P(B)

= 0.15/0.7

∴ P(A/B) = 3/14

Further, by addition theorem

P(A∪B) = P(A) + P(B) - P(A∩B)

= 0.3 + 0.7 – 0.15 = 0.85

9.3.3 Multiplication theorem:

Statement: Let S be sample space associated with the given random experiment. Let A and B be any two events defined on the sample space S. Then the probability of occurrence of both the events is denoted by P(A∩B) and is given by

P(A∩B) = P(A).P(B/A)

Since $P(B/A) = \frac{P(A\cap B)}{P(A)}$

∴ P(A∩B) = P(A).P(B/A)

Similarly P(A∩B) = P(B).P(A/B)

SOLVED EXAMPLES

Ex. 1) Two cards are drawn from a pack of 52 cards one after other without replacement. What is the probability that both cards are ace cards?

Soln.: Let event A: first card drawn is an Ace card.

Let event B: second card drawn is an Ace card.

∴ required probability = P(both are Ace cards)

= P(A∩B) = P(A)P(B/A)

Now $P(A) = \frac{4}{52} = \frac{1}{13}$

Since first ace card is not replaced in the pack, therefore now we have 51 cards containing 3 ace cards

∴ Probability of getting second are card under the condition that first ace card is not replaced in the pack = $P(B/A) = \frac{3}{51} = \frac{1}{17}$

∴ P(both are ace cards) = P(A∩B)

$= P(A)P(B/A) = \frac{1}{13} \times \frac{1}{17} = \frac{1}{221}$

Ex. 2) An urn contains 4 black and 6 white balls. Two balls are drawn one after the other without replacement, what is the probability that both balls are black?

Solution : Let event A: first ball drawn in black.

Event B: second ball drawn is black.

∴ Required probability = P(both are black balls)

P(A∩B) = P(A)P(B/A)

Now $P(A) = \frac{4}{10}$

Since first black ball is not replaced in the urn, therefore now we have 9 balls containing 3 black balls.

$\therefore$ Probability of getting second black ball under the condition that first black is not replaced in the pack = $P(B/A) = \frac{3}{9}$

$\therefore$ P(both are black balls) = $P(A \cap B)$

$= P(A)P(B/A) = \frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$

9.3.4 Independent Events:

Let S be sample space associated with the given random experiment. Let A and B be any two events defined on the sample space S. If the occurrence of either event, does not affect the probability of the occurence of the other event, then the two events A and B are said to be independent.

Thus, if A and B are independent events then, $P(A/B) = P(A/B') = P(A)$ and

$P(B/A) = P(B/A') = P(B)$

Remark: If A and B are independent events then $P(A \cap B) = P(A).P(B)$

note that $P(A \cap B) = P(A).(B/A)$

$= P(A).P(B)$

$\therefore$ **$P(A \cap B) = P(A).P(B)$**

In general, if $A_1, A_2, A_3, \ldots A_n$ are n mutually independent events, then

$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1).P(A_2)\ldots P(A_n)$

Theorem:

If A and B are independent events then

a) A and B' are also independent event

b) A' and B' are also independent event

Proof: Since A and B are independent, therefore $P(A \cap B) = P(A).P(B) \ldots (1)$

a) $P(A \cap B') = P(A) - P(A \cap B)$

$= P(A) - P(A).P(B)$ [From (1)]

$= P(A)\{1 - P(B)\}$

$= P(A).P(B')$

$\therefore$ A and B' are also independent.

b) $P(A' \cap B') = P(A \cup B)'$

(By De Morgan's Law)

$= 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - P(A) - P(B) + P(A).P(B)$ from (I)

$= [1 - P(A)] - P(B)[1 - P(A)]$

$= [1 - P(A)]\ [1 - P(B)]$

$= P(A').P(B')$

$\therefore$ A' and B' are also independent.

SOLVED EXAMPLES

Ex. 1: Two cards are drawn at random one after the other. Given that first card drawn in non-face red card, what is the probability that second card is face card, if the cards are drawn

i) without replacement? ii) with replacement?

Solution : Let event A: first card drawn is a non-face red card and event B: second card drawn is face card.

$\therefore P(A) = \frac{20}{52} = \frac{5}{13}$ and $P(B) = \frac{12}{52} = \frac{3}{13}$

$\therefore$ required probability = P(second card drawn is face card given that it is a red card)

i) Without replacement: Since first non-face red card is not replaced, therefore now we have 51 cards containing 12 face cards.

$\therefore P(B/A) = \frac{12}{51} \neq P(B)$. In this case A and B are not independent.

ii) With replacement: Since first non-face red card is replaced, therefore now again we have 52 cards containing 12 face cards.

$\therefore$ P(B/A) = $\frac{12}{52} = \frac{3}{13}$ = P(B).

In this case A and B are independent.

Ex.2: If A and B are two independent events and P(A) = $\frac{3}{5}$, P(B) = $\frac{2}{3}$, find

i) P(A∩B) ii) P(A∩B') iii) P(A'∩B) iv) P(A'∩B') v) P(A∪B)

Solution : P(A) = $\frac{3}{5}$ $\therefore$P(A') = 1 -P(A) = $\frac{2}{5}$

P(B) = $\frac{2}{3}$ $\therefore$P(B') = 1 – P(B) = $\frac{1}{3}$

i) P(A∩B) = P(A)P(B) = $\frac{2}{5}$

ii) P(A∩B') =P(A)P(B') = $\frac{1}{5}$

iii) P(A'∩B) =P(A')P(B) = $\frac{4}{15}$

iv) P(A'∩B') =P(A')P(B') = $\frac{2}{15}$

v) P(A∪B) = P(A) + P(B) - P(A∩B) = $\frac{13}{15}$

Ex. 3) Three professors A, B and C appear in an interview for the post of Principal. Their chances of getting selected as a principal are $\frac{2}{9}$, $\frac{4}{9}$, $\frac{1}{3}$. The probabilities they introduce new course in the college are $\frac{3}{10}$, $\frac{1}{2}$, $\frac{4}{5}$ respectively. Find the probability that the new course is introduced.

Solution : Let A, B, C be the events that prof. A, B and C are selected as principal.

Given P(A) = $\frac{2}{9}$, P(B) = $\frac{4}{9}$, P(C) = $\frac{1}{3} = \frac{3}{9}$

Let N be the event that New Course in introduced P(N/A) = $\square$, P(N/B) = $\square$, P(N/C) = $\boxed{\frac{4}{5}}$

N = (A∩N) ∪ (B∩$\square$) ∪ ($\square$∩N)

$\therefore$ P(N) = P(A∩N) + P(B∩$\square$) + P($\square$∩N)

= P(A).P(N/A) + P$\square$ × $\square$

+ $\square$ × P(N/C)

= $\square$ $\square$ + $\square$ $\square$ + $\square$ $\square$

= $\square$ + $\square$ + $\square$ = $\square$

EXERCISE 9.3

1) A bag contains 3 red marbles and 4 blue marbles. Two marbles are drawn at random without replacement. If the first marble drawn is red, what is the probability the second marble is blue?

2) A box contains 5 green pencils and 7 yellow pencils. Two pencils are chosen at random from the box without replacement. What is the probability that both are yellow?

3) In a sample of 40 vehicles, 18 are red, 6 are trucks, of which 2 are red. Suppose that a randomly selected vehicle is red. What is the probability it is a truck?

4) From a pack of well-shuffled cards, two cards are drawn at random. Find the probability that both the cards are diamonds when

   i) first card drawn is kept aside

   ii) the first card drawn is replaced in the pack.

5) A, B, and C try to hit a target simultaneously but independently. Their respective probabilities of hitting the target are $\frac{3}{4}$, $\frac{1}{2}$ and $\frac{5}{8}$. Find the probability that the target

   a) is hit exactly by one of them

   b) is not hit by any one of them

   c) is hit

   d) is exactly hit by two of them

6) The probability that a student X solves a problem in dynamics is $\frac{2}{5}$ and the probability that student Y solves the same problem is $\frac{1}{4}$. What is the probability that

i) the problem is not solved

ii) the problem is solved

iii) the problem is solved exactly by one of them

7) A speaks truth in 80% of the cases and B speaks truth in 60% of the cases. Find the probability that they contradict each other in narrating an incident.

8) Two hundred patients who had either Eye surgery or Throat surgery were asked whether they were satisfied or unsatisfied regarding the result of their surgery.

The follwoing table summarizes their response.

Surgery	Satisfied	Unsatisfied	Total
Throat	70	25	95
Eye	90	15	105
Total	160	40	200

If one person from the 200 patients is selected at random, determine the probability

a) that the person was satisfied given that the person had Throat surgery

b) that person was unsatisfied given that the person had eye surgery

c) the person had Throat surgery given that the person was unsatisfied

9) Two dice are thrown together. Let A be the event 'getting 6 on the first die' and B be the event 'getting 2 on the second die'. Are the events A and B independent?

10) The probability that a man who is 45 years old will be alive till he becomes 70 is $\frac{5}{12}$. The probability that his wife who is 40 years old will be alive till she becomes 65 is $\frac{3}{8}$. What is the probability that, 25 years hence,

a) the couple will be alive

b) exactly one of them will be alive

c) none of them will be alive

d) at least one of them will be alive

11) A box contains 10 red balls and 15 green balls. Two balls are drawn in succession without replacement. What is the probability that,

a) the first is red and the second is green?

b) one is red and the other is green?

12) A bag contains 3 yellow and 5 brown balls. Another bag contains 4 yellow and 6 brown balls. If one ball is drawn from each bag, what is the probability that,

a) both the balls are of the same color?

b) the balls are of different color?

13) An urn contains 4 black, 5 white and 6 red balls. Two balls are drawn one after the other without replacement. What is the probability that at least one of them is black?

14) Three fair coins are tossed. What is the probability of getting three heads given that at least two coins show heads?

15) Two cards are drawn one after the other from a pack of 52 cards without replacement. What is the probability that both the cards drawn are face cards?

16) Bag A contains 3 red and 2 white balls and bag B contains 2 red and 5 white balls. A bag is selected at random, a ball is drawn and put into the other bag, and then a ball is drawn from that bag. Find the probability that both the balls drawn are of same color.

17) (Activity) : A bag contains 3 red and 5 white balls. Two balls are drawn at random one after the other without replacement. Find the probability that both the balls are white.

Solution : Let,

A : First ball drawn is white

B : second ball drawn in white.

$P(A) = \frac{\square}{\square}$

After drawing the first ball, without replacing it into the bag a second ball is drawn from the remaining $\square$ balls.

$\therefore$ $P(B/A) = \frac{\square}{\square}$

$\therefore$ P(Both balls are white) = $P(A \cap B)$

$= P(\square).P(\square)$

$= \square \quad \square$

$= \square$

18) A family has two children. Find the probability that both the children are girls, given that atleast one of them is a girl.

9.4 Bayes' Theorem:

(This is also known as Bayes' Law and sometimes Bayes' Rule). This is a direct application of conditional probabilities. Bayes' theorem is useful, to determine posterior probabilities.

Theorem : If $E_1, E_2, E_3 \ldots E_n$ are mutually exclusive and exhaustive events with $P(E_i) \neq 0$, where $i = 1, 2, 3 \ldots n$ then for any arbitrary event A which is a subset of the union of events E_i such that $P(A) > 0$, we have

$$P(E_i/A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i).P(A/E_i)}{\sum_{i=1}^{n} P(A \cap E_i)}$$

Proof : We have $A = (A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \ldots \cup (A \cap E_n)$

$A \cap E_1$, $A \cap E_2$, $A \cap E_3 \ldots A \cap E_n$ are mutually exclusive events

So, $P(A) = P[(A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \ldots \cup (A \cap E_n)]$

$$P(A) = \sum_{i=1}^{n} P(A \cap E_i)$$

Also, $P(A \cap E_i) = P(A).P(E_i/A)$

i.e. $P(E_i/A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i).P(A/E_i)}{\sum_{i=1}^{n} P(A \cap E_i)}$

Three types of probabilities occur in the above formula $P(E_i)$, $P(A/E_i)$, $P(E_i/A)$

i) The probabilities occur in the above formula $P(E_i)$, $i = 1, 2, 3, \ldots n$ are such that $P(E_1) + P(E_2) + \ldots + P(E_n) = 1$ are called prior probabilities, since they are known before conducting experiment.

ii) The probabilities $P(A/E_i)$ tell us, how likely the event A under consideration occurs, given each and every prior probability. They may refer to as likelihood probabilities of the event A, given that event Ei has already occurred.

iii) The conditional probabilities $P(E_i/A)$ are called posterior probabilities, as they obtained after conducting experiment.

Bayes' theorem for n = 3 is explained in the following figure.

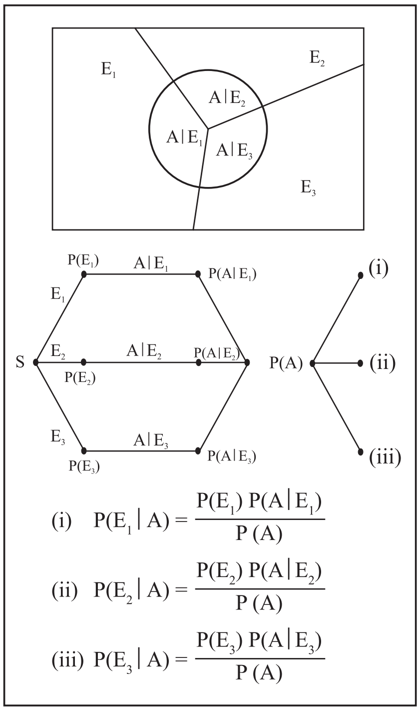


Fig. 9.2 (a) & (b)

SOLVED EXAMPLES

Ex. 1: A bag contains 6 red, 5blue balls and another bag contains 5 red and 8 blue balls. A ball is drawn from first bag and without noticing colour is put in the second bag. A ball is drawn from the second bag. Find the probability that ball drawn is blue in colour.

Solution: Let event E_1: Red ball is drawn from the first bag and event E_2: Blue ball is drawn from the first bag.

$\therefore P(E_1) = 6/11$ and $P(E_2) = \frac{5}{11}$ (Note that E_1 and E_2 are mutually exclusive and exhaustive events)

Let event A: Blue ball is drawn from the second bag

$\therefore P(A/E_1)$ = P (Blue ball is drawn from the second under the condition that red ball is transferred from first bag to second bag) $= \frac{8}{14}$

Similarly, $P(A/E_2)$ = P (Blue ball is drawn from the second under the condition that blue ball is transferred from first bag to second bag) $= \frac{9}{14}$

$\therefore$ required probability = P(Blue ball is drawn from the second bag)

$\therefore P(A) = P(A \cap E_1) + P(A \cap E_2)$

$= P(E_1)P(A/E_1) + P(E_2)P(A/E_2)$

$$= \left(\frac{6}{11}\right)\left(\frac{8}{14}\right) + \left(\frac{5}{11}\right)\left(\frac{9}{14}\right)$$

$$= \left(\frac{48}{154}\right)\left(\frac{45}{154}\right) = \frac{93}{154}$$

Ex. 2: The chances of X, Y, Z becoming managers of a certain company are 4:2:3. The probabilities that the bonus scheme will be introduced if X, Y, Z become managers are 0.3, 0.5 and 0.8 respectively. If the bonus scheme has been introduced, what is the probability that X is appointed as the manger?

Solution: Let E1: Person X becomes manager

E2: Person Y becomes manager

Let E3: Person Z becomes manager

$\therefore P(E_1) = \frac{4}{9}$; $P(E_2) = \frac{2}{9}$; $P(E_3) = \frac{3}{9}$

(Note that E_1, E_2 and E_3 are mutually exclusive and exhaustive events)

Let event A: Bonus is introduced.

$\therefore P(A/E_1)$ = P(Bonus is introduced under the condition that person X becomes manager) = 0.3

$P(A/E_2)$ = P(Bonus is introduced under the condition that person Y becomes manager) = 0.5

and $P(A/E_3)$ = P(Bonus is introduced under the condition that person Z becomes manager) = 0.8

$\therefore P(A) = P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3)$

$= P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)$ $P(A/E_3)$

$$= \left(\frac{4}{9}\right)(0.3) + \left(\frac{2}{9}\right)(0.5) + \left(\frac{3}{9}\right)(0.8)$$

$$= \frac{23}{45}$$

$\therefore$ required probability = P(Person X becomes manager under the condition that bonus scheme is introduced)

$= P(E_1/A) = P(A \cap E_1) / P(A)$

$$= \frac{(2/15)}{(23/45)}$$

$$= \frac{6}{23}$$

Ex. 3: The members of the consulting firm hire cars from three rental agencies, 60% from agency X, 30% from agency Y and 10% from agency Z. 9% of the cars from agency X need repairs, 20% of the cars from agency Y need repairs and 6% of the cars from agency Z need repairs. If a rental car delivered to the consulting firms needs repairs, what is the probability that it came from rental agency Y?

Solution : If A is the event that the car needs repairs and B, C, D are the events that the car comes from rental agencies X, Y or Z. We have P(B) = 0.6, P(C) = 0.3, P(D) = 0.1, P(A/B) = 0.09, P(A/C) = 0.2 and P(A/D) = 0.06

$P(A) = P(A \cap B) + P(A \cap C) + P(A \cap D)$

$= P(B).P(A/B) + P(C).P(A/C) + P(D).P(A/D)$

$= 0.6 \times 0.09 + 0.3 \times 0.2 + 0.1 \times 0.06$

$= 0.054 + 0.06 + 0.006$

$\therefore\ P(A) = 0.12$

$$P(C/A) = \frac{P(A \cap C)}{P(A)} = \frac{P(C).P(A/C)}{P(A)}$$

$$= \frac{0.3 \times 0.2}{0.12} = \frac{0.06}{0.12}$$

$= 0.5$

EXERCISE 9.4

1) There are three bags, each containing 100 marbles. Bag 1 has 75 red and 25 blue marbles. Bag 2 has 60 red and 40 blue marbles and Bag 3 has 45 red and 55 blue marbles. One of the bags is chosen at random and a marble is picked from the chosen bag. What is the probability that the chosen marble is red?

2) A box contains 2 blue and 3 pink balls and another box contains 4 blue and 5 pink balls. One ball is drawn at random from one of the two boxes and it is found to be pink. Find the probability that it was drawn from (i) first box (ii) second box.

3) There is a working women's hostel in a town, where 75% are from neighbouring town. The rest all are from the same town. 48% of women who hail from the same town are graduates and 83% of the women who have come from the neighboring town are also graduates. Find the probability that a woman selected at random is a graduate from the same town.

4) If E_1 and E_2 are equally likely, mutually exclusive and exhaustive events and $P(A/E_1) = 0.2$, $P(A/E_2) = 0.3$. Find $P(E_1/A)$.

5) Jar I contains 5 white and 7 black balls. Jar II contains 3 white and 12 black balls. A fair coin is flipped; if it is Head, a ball is drawn from Jar I, and if it is Tail, a ball is drawn from Jar II. Suppose that this experiment is done and a white ball was drawn. What is the probability that this ball was in fact taken from Jar II?

6) A diagnostic test has a probability 0.95 of giving a positive result when applied to a person suffering from a certain disease, and a probability 0.10 of giving a (false) positive result when applied to a non-sufferer. It is estimated that 0.5% of the population are sufferers. Suppose that the test is now administered to a person about whom we have no relevant information relating to the disease (apart from the fact that he/she comes from this population). Calculate the probability that:

   a) given a positive result, the person is a sufferer.

   b) given a negative result, the person is a non-sufferer.

7) A doctor is called to see a sick child. The doctor has prior information that 80% of the sick children in that area have the flu, while the other 20% are sick with measles. Assume that there is no other disease in that area. A well-known symptom of measles is rash. From the past records, it is known

that, chances of having rashes given that sick child is suffering from measles is 0.95. However occasionally children with flu also develop rash, whose chance are 0.08. Upon examining the child, the doctor finds a rash. What is the probability that child is suffering from measles?

8) 2% of the population have a certain blood disease of a serious form: 10% have it in a mild form; and 88% don't have it at all. A new blood test is developed; the probability of testing positive is $\frac{9}{10}$ if the subject has the serious form, $\frac{6}{10}$ if the subject has the mild form, and $\frac{1}{10}$ if the subject doesn't have the disease. A subject is tested positive. What is the probability that the subject has serious form of the disease?

9) A box contains three coins: two fair coins and one fake two-headed coin is picked randomly from the box and tossed.

a) What is the probability that it lands head up?

b) If happens to be head, what is the probability that it is the two-headed coin?

10) There are three social media groups on a mobile: Group I, Group II and Group III. The probabilities that Group I, Group II and Group III sending the messages on sports are $\frac{2}{5}$, $\frac{1}{2}$, and $\frac{2}{3}$ respectively. The probability of opening the messages by Group I, Group II and Group III are $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$ respectively. Randomly one of the messages is opened and found a message on sports. What is the probability that the message was from Group III.

11) (Activity):Mr.XgoestoofficebyAuto,Carand train.Theprobabilitieshimtravellingbythese modes are $\frac{2}{7}$, $\frac{3}{7}$, $\frac{2}{7}$ respectively. The chances of him being late to the office are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$ respectively by Auto, Car and train. On one particular day he was late to the office. Find the probability that he travelled by car.

Solution : Let A, C and T be the events that Mr. X goes to office by Auto, Car and Train respectively. Let L be event that he is late.

Given that P(A) = $\square$, P(B) = $\square$,

P(C) = $\square$

P(L/A) = $\frac{1}{2}$, P(L/B) = $\square$, P(L/C) = $\frac{1}{4}$

$P(L) = P(A \cap L) + P(C \cap L) + P(T \cap L)$

$= P(A).P(L/A) + P(C).P(L/C) + P(T).P(L/T)$

$= \square\,\square + \square\,\square + \square\,\square$

$= \square + \square + \square$

$= \square$

$P(L/C) = \frac{P(A \cap C)}{P(L)} = \frac{P(C).P(L/C)}{P(L)}$

$= \frac{\square\,\square}{\square}$

$= \square$

9.5 ODDS (Ratio of two complementary probabilities):

Let n be number of distinct sample points in the sample space S. Out of n sample points, m sample points are favourable for the occurrence of event A. Therefore remaining (n-m) sample points are favourable for the occurrence of its complementary event A'.

$\therefore P(A) = \frac{m}{n}$ and $P(A') = \frac{n-m}{n}$

Ratio of number of favourable cases to number of unfavourable cases is called as odds in favour of event A which is given by $\frac{m}{n-m}$ i.e. P(A):P(A')

Ratio of number of unfavourable cases to number of favourable cases is called as odds against event A which is given by $\frac{n-m}{m}$ i.e. P(A'):P(A)

SOLVED EXAMPLES

Ex. 1: A fair die is thrown. What are the odds in favour of getting a number which is a perfect square in uppermost face of die?

Soln.: Random experiment: A fair die is thrown.

$\therefore$ Sample space S = {1, 2, 3, 4, 5, 6}

$\therefore$ n(S) = 6

Let event A: die shows number which is a perfect square.

$\therefore A = \{1, 4\}$ $\therefore m = n(A) = 2$

$\therefore A' = \{2, 3, 5, 6\}$ $\therefore (n - m) = 4$.

$$\therefore P(A) = \frac{m}{n} = \frac{2}{6} = \frac{1}{3}$$

$$P(A') = \frac{n-m}{n} = \frac{4}{6} = \frac{2}{3}$$

$\therefore$ Odds in favour of event

$$A = P(A) : P(A') = \frac{1/3}{2/3} = \frac{1}{2}$$

Ex. 2: The probability of one event A happening is the square of the probability of second event B, but the odds against the event A are the cube of the odds against the event B. Find the probability of each event.

Solution : Let P(A) = p_1 and P(B) = p_2.

$\therefore$ probability on non-occurrence of the events A and B are $(1–p_1)$ and $(1–p_2)$ respectively. We are given that $p_1 = (p_2)^2$ .... (I)

Odds against the event A = $\dfrac{1-p_1}{p_1}$

Odds against the event B = $\dfrac{1-p_2}{p_2}$

Since odds against the event A are the cube of the odds against the event B.

$$\frac{1-p_1}{p_1} = \left(\frac{1-p_2}{p_2}\right)^3$$

$$\frac{1-p_2^2}{p_2^2} = \frac{(1-p_2)^3}{p_2^3} \qquad \text{[By (I)]}$$

$$\frac{(1-p_2)(1+p_2)}{1} = \frac{(1-p_2)^3}{p_2}$$

$\therefore p_2(1+p_2) = 1-2p_2 + p_2^2$

$p_2+p_2^2 = 1-2p_2 + p_2^2$

$3p_2 = 1$

$\therefore p_2 = \dfrac{1}{3}$

$p_1 = (p_2)^2 = \left(\dfrac{1}{3}\right)^2 = \dfrac{1}{9}$

$\therefore P(A) = \dfrac{1}{9}$ and $P(B) = \dfrac{1}{3}$

EXERCISE 9.5

1) If odds in favour of X solving a problem are 4:3 and odds against Y solving the same problem are 2:3. Find probability of:

   i) X solving the problem

   ii) Y solving the problem

2) The odds against John solving a problem are 4 to 3 and the odds in favor of Rafi solving the same problem are 7 to 5. What is the chance that the problem is solved when both them try it?

3) The odds against student X solving a statistics problem are 8:6 and odds in favour of student y solving the same problem are 14:16. Find is the chance that

   i) the problem will be solved if they try it independently

   ii) neither of them solves the problem

4) The odds against a husband who is 60 years old, living till he is 85 are 7:5. The odds against his wife who is now 56, living till she is 81 are 5:3. Find the probability that

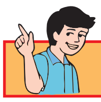
   a) at least one of them will be alive 25 years hence

   b) exactly one of them will be alive 25 years hence.

5) There are three events A, B and C, one of which must, and only one can happen. The odds against the event A are 7:4 and odds against event B are 5:3. Find the odds against event C.

6) In a single toss of a fair die, what are the odds against the event that number 3 or 4 turns up?

7) The odds in favour of A winning a game of chess against B are 3:2. If three games are to be played, what are the odds in favour of A's winning at least two games out of the three?



Let's Remember

A.N. Kolmogorov, a Russian mathematician outlined an axiomatic definition of probability that formed the basis of the modern theory. For every event A of sample space S, we assign a non-negative real number denoted by P(A) and is called probability of A, which satisfied following three axioms

1) $0 \leq P(A) \leq 1$

2) $P(S) = 1$

If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$

Addition theorem:

If A and B are any two events defined on the same sample space S, then probability of occurrence of at least one event is denoted by $P(A \cup B)$ and is given by $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Conditional probability:

If A and B are any two events defined on the same sample space S, then conditional probability of event A given that event B has already occurred is denoted by P(A/B)

$\therefore P(A/B) = P(A \cap B) / P(B),\ P(B) \neq 0$

Similarly, $P(B/A) = P(A \cap B) / P(A),\ P(A) \neq 0$

Multiplication theorem:

If A and B are any two events defined on the same sample space S, then probability of simultaneous occurrence of both events is denoted by $P(A \cap B)$ and is given by $P(A \cap B) = P(A)P(B/A)$

Independent events:

If the occurrence of any one event does not depend on occurrence of other event, then two events A and B are said to be independent.

i.e. if $P(A/B) = P(A/B') = P(A)$

or $P(B/A) = P(B/A') = P(B)$

then A and B are independent events.

$\therefore P(A \cap B) = P(A)P(B)$

If A and B are independent events then

a) A and B' are also independent events

b) A' and B' are also independent events.

Bayes' Theorem :

If $E_1, E_2, E_3 \ldots E_n$ are mutually exclusive and exhaustive events with $P(E_i) \neq 0$, where $i = 1,2,3 \ldots n$. then for any arbitrary event A which is a subset of the union of events E such that $P(A) > 0$, we have

$$P(E_i/A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i).P(A/E_i)}{\sum_{i=1}^{n} P(A \cap E_i)}$$

MISCELLANEOUS EXERCISE - 9

I) Select the correct answer from the given four alternatives.

1) There are 5 girls and 2 boys, then the probability that no two boys are sitting together for a photograph is

A) $\frac{1}{21}$ B) $\frac{4}{7}$ C) $\frac{2}{7}$ D) $\frac{5}{7}$

2) In a jar there are 5 black marbles and 3 green marbles. Two marbles are picked randomly one after the other without replacement.

What is the possibility that both the marbles are black?

A) $\frac{5}{14}$ B) $\frac{5}{8}$ C) $\frac{5}{8}$ D) $\frac{5}{16}$

3) Two dice are thrown simultaneously. Then the probability of getting two numbers whose product is even is

A) $\frac{3}{4}$ B) $\frac{1}{4}$ C) $\frac{5}{7}$ D) $\frac{1}{2}$

4) In a set of 30 shirts, 17 are white and rest are black. 4 white and 5 black shirts are tagged as 'PARTY WEAR'. If a shirt is chosen at random from this set, the possibility of choosing a black shirt or a 'PARTY WEAR' shirt is

A) $\frac{11}{15}$ B) $\frac{13}{30}$ C) $\frac{9}{13}$ D) $\frac{17}{30}$

5) There are 2 shelves. One shelf has 5 Physics and 3 Biology books and the other has 4 Physics and 2 Biology books. The probability of drawing a Physics book is

A) $\frac{9}{14}$ B) $\frac{31}{48}$ C) $\frac{9}{38}$ D) $\frac{1}{2}$

6) Two friends A and B apply for a job in the same company. The chances of A getting selected is 2/5 and that of B is 4/7. The probability that both of them get selected is

A) $\frac{34}{35}$ B) $\frac{1}{35}$ C) $\frac{8}{35}$ D) $\frac{27}{35}$

7) The probability that a student knows the correct answer to a multiple choice question is $\frac{2}{3}$. If the student does not know the answer, then the student guesses the answer. The probability of the guessed answer being correct is $\frac{1}{4}$. Given that the student has answered the question correctly, the probability that the student knows the correct answer is

A) $\frac{5}{6}$ B) $\frac{6}{7}$ C) $\frac{7}{8}$ D) $\frac{8}{9}$

8) Bag I contains 3 red and 4 black balls while another Bag II contains 5 red and 6 black balls. One ball is drawn at random from one of the bags and it is found to be red. The probability that it was drawn from Bag II.

A) $\frac{33}{68}$ B) $\frac{35}{69}$ C) $\frac{34}{67}$ D) $\frac{35}{68}$

9) A fair is tossed twice. What are the odds in favour of getting 4, 5 or 6 on the first toss and 1, 2, 3 or 4 on the second die?

A) 1 : 3 B) 3 : 1 C) 1 : 2 D) 2 : 1

10) The odds against an event are 5:3 and the odds in favour of another independent event are 7:5. The probability that at least one of the two events will occur is

A) $\frac{52}{96}$ B) $\frac{71}{96}$ C) $\frac{69}{96}$ D) $\frac{13}{96}$

II) Solve the following.

1) The letters of the word 'EQUATION' are arranged in a row. Find the probability that a) All the vowels are together b) Arrangement starts wiht a vowel and ends wiith a consonant.

2) There are 6 positive and 8 negative numbers. Four numbers are chosen at random, without replacement, and multiplied. Find the probability that the product is a positive numbers.

3) Ten cards numbered 1 to 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is and even number?

4) If $P(A \cap B) = \frac{1}{2}$, $P(B \cap C) = \frac{1}{3}$, $P(C \cap A) = \frac{1}{6}$ then find P(A), P(B) and P(C), If A,B,C are independent events.

5) If the letters of the word 'REGULATIONS' be arranged at random, what is the probability that there will be exactly 4 letters between R and E?

6) In how many ways can the letters of the word ARRANGEMENTS be arranged?

a) Find the chance that an arrangement chosen at random begins with the letters EE.

b) Find the probability that the consonants are together.

7) A letter is taken at random from the letters of the word 'ASSISTANT' and another letter is taken at random from the letters of the word 'STATISTICS'. Find probability that the selected letters are the same.

8) A die is loaded in such a way that the probability of the face with j dots turning up is proportional to j for j = 1, 2, ...6. What is the probability, in one roll of the die, that an odd number of dots will turn up?

9) An urn contains 5 red balls and 2 green balls. A ball is drawn. If it's green a red ball is added to the urn and if it's red a green ball is added to the urn. (The original ball is not returned to the urn). Then a second ball is drawn. What is the probability the second ball is red?

10) The odds against A solving a certain problem are 4 to 3 and the odds in favor of solving the same problem are 7 to 5 find the probability that the problem will be solved.

11) If $P(A) = P\left(\frac{A}{B}\right) = \frac{1}{5}$, $P\left(\frac{B}{A}\right) = \frac{1}{3}$ then find

(i) $P\left(\frac{A'}{B}\right)$ (ii) $P\left(\frac{B'}{A'}\right)$

12) Let A and B be independent events with $P(A) = \frac{1}{4}$, and $P(A \cup B) = 2P(B) - P(A)$.

Find a) P(B); b) P(A/B); and c) P(B'/A).

13) Find the probability that a year selected will have 53 Wednesdays.

14) The chances of P, Q and R, getting selected as principal of a college are $\frac{2}{5}, \frac{2}{5}, \frac{1}{5}$ respectively. Their chances of introducing IT in the college are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. Find the probability that

a) IT is introduced in the college after one of them is selected as a principal .

b) IT is introduced by Q.

15) Suppose that five good fuses and two defective ones have been mixed up. To find the defective fuses, we test them one-by-one, at random and without replacement. What is the probability that we are lucky and fine both of the defective fuses in the first two tests?

16) Fot three events A, B and C, we know that A and C are independent, B and C are independent, A and B are disjoint, $P(A \cup C) = 2/3$, $P(B \cup C) = 3/4$, $P(A \cup B \cup C) = 11/12$. Find P(A), P(B) and P(C).

17) The ratio of Boys to Girls in a college is 3:2 and 3 girls out of 500 and 2 boys out of 50 of that college are good singers. A good singer is chosen what is the probability that the chosen singer is a girl?

18) A and B throw a die alternatively till one of them gets a 3 and wins the game. Find the respective probabilities of winning. (Assuming A begins the game).

19) Consider independent trails consisting of rolling a pair of fair dice, over and over What is the probability that a sum of 5 appears before sum of 7?

20) A machine produces parts that are either good (90%), slightly defective (2%), or obviously defective (8%). Produced parts get passed through an automatic inspection machine, which is able to detect any part that is obviously defective and discard it. What is the quality of the parts that make it throught the inspection machine and get shipped?

21) Given three identical boxes, I, II and III, each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver soin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?

22) In a factory which manufactures bulbs, machines A, B and C manufacture respectively 25%, 35% and 40% of the bulbs. Of their outputs, 5, 4 and 2 percent are respectively defective bulbs. A bulbs is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the machine B?

23) A family has two children. One of them is chosen at random and found that the child is a girl. Find the probability that

a) both the children are girls.

b) both the children are girls given that at least one of them is a girl.