4 Determinants and Matrices

Let's Study

- Definition and Expansion of Determinants
- Minors and Co-factors of determinants
- Properties of Determinants
- Applications of Determinants
- Introduction and types of Matrices
- Operations on Matrices
- Properties of related matrices

4.1 Introduction

We have learnt to solve simultaneous equations in two variables using determinants. We will now learn more about the determinants because they are useful in Engineering applications, and Economics, etc.

> The concept of a determinant was discussed by the German Mathematician G.W. Leibnitz (1676-1714) and Cramer (1750) developed the rule for solving linear equations using determinants.

Let's Recall

4.1.1 Value of a Determinant

In standard X we have studied a method of solving simultaneous equations in two unknowns using determinants of order two. In this chapter, we shall study determinants of order three.

The representation $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is defined as the determinant of order two. Numbers a, b, c, d are called elements of the determinant. In this arrangement, there are two rows and two columns.

$$\begin{array}{cc} \begin{vmatrix} a & b \\ c & d \end{vmatrix} & \begin{array}{l} \leftarrow 1^{st} \text{ row} \\ \leftarrow 2^{nd} \text{ row} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ 1^{st} \text{ column} & 2^{nd} \text{ column} \end{array} & \end{array}$$

The value of the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is $ad - bc$.

SOLVED EXAMPLES

Ex. Evaluate i) $\begin{vmatrix} 7 & 9 \\ -4 & 3 \end{vmatrix}$ ii) $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$

iii) $\begin{vmatrix} 4 & i \\ -2i & 7 \end{vmatrix}$ where $i^2 = -1$ iv) $\begin{vmatrix} \log_4^{\,2} & \log_4^{\,2} \\ 2 & 4 \end{vmatrix}$

Solution :

i) $\begin{vmatrix} 7 & 9 \\ -4 & 3 \end{vmatrix} = 7 \times 3 - (-4) \times 9 = 21 + 36 = 57$

ii) $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta - (-\sin^2\theta)$

$= \cos^2\theta + \sin^2\theta = 1$

iii) $\begin{vmatrix} 4 & i \\ -2i & 7 \end{vmatrix} = 4 \times 7 - (-2i) \times i = 28 + 2i^2$

$= 28 + 2(-1) \quad [\because i^2 = -1]$

$= 28 - 2 = 26$

iv) $\begin{vmatrix} \log_4^{\,2} & \log_4^{\,2} \\ 2 & 4 \end{vmatrix} = 4 \times \log_4 2 - 2 \times \log_4^{\,2}$

$= \log_4 2^4 - \log_4 2^2$

$$= \log_4 16 - \log_4 4$$
$$= 2\log_4 4 - \log_4 4$$
$$= 2 \times 1 - 1 = 2 - 1 = 1$$

Let's Understand

4.1.2 Determinant of order 3

Definition - A determinant of order 3 is a square arrangement of 9 elements enclosed between two vertical bars. The elements are arranged in 3 rows and 3 columns as given below.

$$\begin{array}{cl} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \\ \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} & \end{array}$$

R_i are the rows
C_j are the column

Here a_{ij} represents the element in i^{th} row and j^{th} column of the determinant.

e.g. a_{31} represents the element in 3^{rd} row and 1^{st} column.

In general, we denote determinant by Capital Letters or by Δ (delta).

We can wirte the rows and columns separately. e.g. here the 2^{nd} row is $[a_{21} \ \ a_{22} \ \ a_{23}]$ and 3^{rd} column is

$$\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

Expansion of Determinant

We will find the value or expansion of a 3x3 determinant.We give here the expansion by the 1^{st} row of the determinant D.

There are six ways of expanding a determinant of order 3, corresponding to each of three rows (R_1, R_2, R_3) and three columns (C_1, C_2, C_3).

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The determinant can be expanded as follows:

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

SOLVED EXAMPLES

Evaluate:

i) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & -1 \end{vmatrix}$

ii) $\begin{vmatrix} \sec\theta & \tan\theta & 0 \\ \tan\theta & \sec\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$

iii) $\begin{vmatrix} 2-i & 3 & -1 \\ 3 & 2-i & 0 \\ 2 & -1 & 2-i \end{vmatrix}$ where $i = \sqrt{-1}$

Solution :

i) $$\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & -1 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} - (-4)\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 3(-1+6) + 4(-1+4) + 5(3-2)$$

$$= 3\times5+4\times3+5\times1$$

$$= 15+12+5$$

$$= 32$$

ii) $\begin{vmatrix} \sec\theta & \tan\theta & 0 \\ \tan\theta & \sec\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= \sec\theta \begin{vmatrix} \sec\theta & 0 \\ 0 & 1 \end{vmatrix} - \tan\theta \begin{vmatrix} \tan\theta & 0 \\ 0 & 1 \end{vmatrix}$

$+ 0 \begin{vmatrix} \tan\theta & \sec\theta \\ 0 & 0 \end{vmatrix}$

$= \sec\theta\,(\sec\theta - 0) - \tan\theta\,(\tan\theta - 0) + 0$

$= \sec^2\theta - \tan^2\theta$

$= 1$

iii) $\begin{vmatrix} 2-i & 3 & -1 \\ 3 & 2-i & 0 \\ 2 & -1 & 2-i \end{vmatrix}$

$= (2-i)[(2-i)^2 - 0] - 3[3(2-i) - 0]$
$\quad - 1[-3 - 2(2-i)]$

$= (2-i)^3 - 9(2-i) + 3 + 2(2-i)$

$= 8 - 12i + 6i^2 - i^3 - 18 + 9i + 3 + 4 - 2i$

$= 8 - 12i + 6(-1) + i - 18 + 9i + 3 + 4 - 2i$ (since $i^2 = -1$)

$= 8 - 6 - 18 + 7 - 12i + 9i - 2i + i$

$= -9 - 4i$

Let's Learn

4.1.3 Minors and Cofactors of elements of determinants

Let $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ be a given determinant.

Definitions

The minor of a_{ij} - It is defined as the determinant obtained by eliminating the i^{th} row and j^{th} column of A. That is the row and the column that contain the element a_{ij} are omitted. We denote the minor of the element a_{ij} by M_{ij}

In case of above determinant A

Minor of $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}.a_{33} - a_{32}.a_{23}$

Minor of $a_{12} = M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}.a_{33} - a_{31}.a_{23}$

Minor of $a_{13} = M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}.a_{32} - a_{31}.a_{22}$

Similarly we can find minors of other elements.

Cofactor of a_{ij} -

cofactor of $a_{ij} = (-1)^{i+j}$ minor of $a_{ij} = C_{ij}$

$\therefore$ Cofactor of element $a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}$

The same definition can also be given for elements in 2×2 determinant. Thus in $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

The minor of a is d.

The minor of b is c.

The minor of c is b.

The minor of d is a

SOLVED EXAMPLES

Ex. 1) Find Minors and Cofactors of the elements of determinant

i) $\begin{vmatrix} 2 & -3 \\ 4 & 7 \end{vmatrix}$

Solution : Here $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 2 & -3 \\ 4 & 7 \end{vmatrix}$

$M_{11} = 7$

$C_{11} = (-1)^{1+1} M_{11} = (-1)^{1+1}.7 = 7$

$M_{12} = 4$

$C_{12} = (-1)^{1+2} M_{12} = (-1)^{1+2} .4 = -4$

$M_{21} = -3$

$C_{21} = (-1)^{2+1} M_{21} = (-1)^{2+1} . (-3) = 3$

$M_{22} = 2$

$C_{22} = (-1)^{2+2} M_{22} = (-1)^{2+2} . 2 = 2$

ii) $\begin{vmatrix} 1 & 2 & -3 \\ -2 & 0 & 4 \\ 5 & -1 & 3 \end{vmatrix}$

Solution :

Here $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 2 & -3 \\ -2 & 0 & 4 \\ 5 & -1 & 3 \end{vmatrix}$

$M_{11} = \begin{vmatrix} 0 & 4 \\ -1 & 3 \end{vmatrix} = 0 + 4 = 4$

$C_{11} = (-1)^{1+1} M_{11} = (-1)^{1+1} . 4 = 4$

$M_{12} = \begin{vmatrix} -2 & 4 \\ 5 & 3 \end{vmatrix} = -6 -20 = -26$

$C_{12} = (-1)^{1+2} M_{12} = (-1)^{1+2} .(-26) = 26$

$M_{13} = \begin{vmatrix} -2 & 0 \\ 5 & -1 \end{vmatrix} = 2 - 0 = 2$

$C_{13} = (-1)^{1+3} M_{13} = (-1)^{1+3} . 2 = 2$

$M_{21} = \begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = 6 - 3 = 3$

$C_{21} = (-1)^{2+1} M_{21} = (-1)^{2+1} . 3 = -3$

$M_{22} = \begin{vmatrix} 1 & -3 \\ 5 & 3 \end{vmatrix} = 3+15 = 18$

$C_{22} = (-1)^{2+2} M_{22} = (-1)^{1+1} . 18 = 18$

$M_{23} = \begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix} = -1 -10 = -11$

$C_{23} = (-1)^{2+3} M_{23} = (-1)^{2+3} . (-11) = 11$

$M_{31} = \begin{vmatrix} 2 & -3 \\ 0 & 4 \end{vmatrix} = 8 - 0 = 8$

$C_{31} = (-1)^{3+1} M_{31} = (-1)^{3+1} . 8 = 8$

$M_{32} = \begin{vmatrix} 1 & -3 \\ -2 & 4 \end{vmatrix} = 4 - 6 = -2$

$C_{32} = (-1)2^{3+2} M_{32} = (-1)^{3+2} . (-2) = 2$

$M_{33} = \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = 0+4 = 4$

$C_{33} = (-1)^{3+3} M_{33} = (-1)^{3+3} . 4 = 4$

Expansion of determinant by using Minor and cofactors of any row/column

$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ (By 1st row)

$= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$ (By 2nd column)

Ex. 2) Find value of x if

i) $\begin{vmatrix} x & -1 & 2 \\ 2x & 1 & -3 \\ 3 & -4 & 5 \end{vmatrix} = -10$

$\therefore\ x(5 - 12) - (-1)(10x + 9) + 2(-8x - 3) = -10$

$\therefore\ x(-7) + (10x + 9) - 16x - 6 = -10$

$\therefore\ -7x + 10x - 16x + 9 - 6 + 10 = 0$

$\therefore\ 10x - 23x + 13 = 0$

$\therefore\ 13x = 13$

$\therefore\ x = 1$

ii) $\begin{vmatrix} x & 3 & 2 \\ x & x & 1 \\ 1 & 0 & 1 \end{vmatrix} = 9$

$\therefore\ x(x-0) - 3(x-1) + 2(0-x) = 9$

$\therefore\ x^2 - 3x + 3 - 2x = 9$

$\therefore\ x^2 - 5x + 3 = 9$

$\therefore\ x^2 - 5x - 6 = 0$

$\therefore\ (x-6)(x+1) = 0$

$\therefore\ x-6 = 0$ or $x+1 = 0$

$\therefore\ x = 6$ or $x = -1$

Ex. 3) Find the value of $\begin{vmatrix} 1 & -1 & 2 \\ -2 & 3 & 5 \\ -2 & 0 & -1 \end{vmatrix}$ by expanding along a) 2nd row b) 3rd column and Interprete the result.

a) Expansion along the 2nd row

$= a_{21}\,c_{21} + a_{22}\,c_{22} + a_{23}\,c_{23}$

$= -2(-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} + 3(-1)^{2+2}\begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} + 5(-1)^{2+3}\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix}$

$= 2(+1-0) + 3(-1+4) - 5(0-2)$

$= 2(1) + 3(3) - 5(-2)$

$= 2 + 9 + 10$

$= 21$

b) Expansion along 3rd coloumn

$= a_{13}\,c_{13} + a_{23}\,c_{23} + a_{33}\,c_{33}$

$= 2(-1)^{1+3}\begin{vmatrix} -2 & 3 \\ -2 & 0 \end{vmatrix} + 5(-1)^{2+3}\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} + $

$- 1(-1)^{3+3}\begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix}$

$= 2(0+6) - 5(0-2) - 1(3-2)$

$= 2(6) - 5(-2) - 1(1)$

$= 12 + 10 - 1$

$= 22 - 1$

$= 21$

Interpretation: From (a) and (b) it is seen that the expansion of determinant by both ways gives the same value.

EXERCISE 4.1

Q.1) Find the value of determinant

i) $\begin{vmatrix} 2 & -4 \\ 7 & -15 \end{vmatrix}$ ii) $\begin{vmatrix} 2i & 3 \\ 4 & -i \end{vmatrix}$ iii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

iv) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Q.2) Find the value of x if

i) $\begin{vmatrix} x^2-x+1 & x+1 \\ x+1 & x+1 \end{vmatrix} = 0$ ii) $\begin{vmatrix} x & -1 & 2 \\ 2x & 1 & -3 \\ 3 & -4 & 5 \end{vmatrix} = 29$

Q.3 Find x and y if $\begin{vmatrix} 4i & i^3 & 2i \\ 1 & 3i^2 & 4 \\ 5 & -3 & i \end{vmatrix} = x+iy$ where $i^2 = -1$

Q.4) Find the minor and cofactor of element of the determinant

$D = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ 5 & 7 & 2 \end{vmatrix}$

Q.5) Evaluate A = $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ Also find minor and cofactor of elements in the 2^{nd} row of determinant and verify

a) $-a_{21}.M_{21} + a_{22}.M_{22} - a_{23}.M_{23}$ = value of A

b) $a_{21}C_{21} + a_{22}C_{22} + a_{23}.C_{23}$ = value of A

where M_{21}, M_{22}, M_{23} are minor of a_{21}, a_{22}, a_{23} and C_{21}, C_{22}, C_{23} are cofactor of a_{21}, a_{22}, a_{23}

Q.6) Find the value of determinant expanding along third column

$$\begin{vmatrix} -1 & 1 & 2 \\ -2 & 3 & -4 \\ -3 & 4 & 0 \end{vmatrix}$$

4.2 Properties of Determinants

In the previous section we have learnt how to expand the determinant. Now we will study some properties of determinants. They will help us to evaluate the determinant more easily.

• Let's Verify…

Property 1 - The value of determinant remains unchanged if its rows are turned into columns and columns are turned into rows.

Verification:

Let D = $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= a_1 .(b_2 c_3 - b_3 c_2) - b_1 (a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$ ---------- (i)

Let D_1 = $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$

$= a_1 (b_2 c_3 - b_3c_2) - a_2(b_1c_3 + c_1b_3) + a_3(b_1c_2 - c_1b_2)$

$= a_1(b_2 c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - b_2a_3)$ ---------- (ii)

From (i) and (ii) D= D_1

Ex.

Let A = $\begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 0 & 2 & 1 \end{vmatrix}$

$= 1\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix}$

$= 1(-1-4) - 2(3-0) - 1(6-0)$

$= -5 - 6 - 6$

$= -17$ …………………………. (i)

by interchanging rows and columns of A we get determinant A_1

$A_1 = \begin{vmatrix} 1 & 3 & 0 \\ 2 & -1 & 2 \\ -1 & 2 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} -3\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix}$

$+ 0\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$

$= 1(-1-4) - 3(2+2) + 0$

$= -5 - 12$

$= -17$ ………………………………..(ii)

$\therefore$ A = A_1 from (i) and (ii)

Property 2 - If any two rows (or columns) of the determinant are interchanged then the value of determinant changes its sign.

The operation $R_i \leftrightarrow R_j$ change the sign of the determinant.

Note : We denote the interchange of rows by $R_i \leftrightarrow R_j$ and interchange of columns by $C_i \leftrightarrow C_j$.

Property 3 - If any two rows (or columns) of a determinant are identical then the value of determinant is zero.

$R_1 \leftrightarrow R_2 \quad D = D_1$

then $D_1 = -D$ .... (property 2) ...... (I)

But $R_1 = R_2$ hence $D_1 = D$ .... (II)

$\therefore$ adding I and II

$2D_1 = 0 \Rightarrow D_1 = 0$

i.e. $D = 0$

Property 4 - If each element of a row (or a column) of determinant is multiplied by a constant k then the value of the new determinant is k times the value of given determinant.

The operation $R_i \to kR_i$ gives multiple of the determinant by k.

Remark i) Using this property we can take out any common factor from any one row (or any one column) of the given determinant

ii) If corresponding elements of any two rows (or columns) of determinant are proportional (in the same ratio) then the value of the determinant is zero.

Property 5 - If each element of a row (or column) is expressed as the sum of two numbers then the determinant can be expressed as sum of two determinants

For example,

$$\begin{vmatrix} a_1+x_1 & b_1+y_1 & c_1+z_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & z_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property 6 - If a constant multiple of all elements of any row (or column) is added to the corresponding elements of any other row (or column) then the value of new determinant so obtained is the same as that of the original determinant. The operation $R_i \leftrightarrow R_i + kR_j$ does not change the value of the determinant.

Verification

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$R_1 \to R_1 + kR_3$

$$A_1 = \begin{vmatrix} a_1+ka_3 & b_1+kb_3 & c_1+kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Simplifying A_1, using the previous properties, we get $A_1 = A$.

Ex. : Let $B = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 1(2–0) – 2(–1–0) + 3(–2–2) = 2 + 2 – 12$

$= 4–12 = –8$ ----------(i)

Now, $B = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \\ 1 & 2 & 1 \end{vmatrix}$

$R_1 \to R_1 + 2R_2$

$$B_1 = \begin{vmatrix} 1+2(-1) & 2+2(2) & 3+2(0) \\ -1 & 2 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$

$B_1 = \begin{vmatrix} -1 & 6 & 3 \\ -1 & 2 & 0 \\ 1 & 2 & 1 \end{vmatrix} = -1(2–0) – 6(–1–0) + 3(–2–2)$

$= –2 + 6 – 12 = 6 – 14 = –8$ ----(ii)

From (i) and (ii) $B = B_1$

Remark : If more than one operation from above are done, make sure that these operations are completed one at a time. Else there can be mistake in calculation.

Main diagonal of determinant : The main diagonal (principal diagonal) of determinant A is collection of entries a_{ij} where $i = j$

OR

Main diagonal of determinant : The set of elements (a_{11} , a_{22}, a_{33} , ---- a_{nn}) is called the main diagonal of the determinant A.

e.g. D = $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ here a_{11}, a_{22} , a_{33} are element of main diagonal

Property 7 - (Triangle property) - If each element of a determinant above or below the main diagonal is zero then the value of the determinant is equal to product of its diagonal elements.

that is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3$$

Remark : If all elements in any row or any column of a determinant are zeros then the value of the determinant is zero.

SOLVED EXAMPLES

Ex. 1) Show that

i) $\begin{vmatrix} 101 & 202 & 303 \\ 505 & 606 & 707 \\ 1 & 2 & 3 \end{vmatrix} = 0$

LHS = $\begin{vmatrix} 101 & 202 & 303 \\ 505 & 606 & 707 \\ 1 & 2 & 3 \end{vmatrix}$

$R_1 \rightarrow R_1 - R_3$

$= \begin{vmatrix} 100 & 200 & 300 \\ 505 & 606 & 707 \\ 1 & 2 & 3 \end{vmatrix}$

$= \begin{vmatrix} 100\times1 & 100\times2 & 100\times3 \\ 505 & 606 & 707 \\ 1 & 2 & 3 \end{vmatrix}$

$= 100 \begin{vmatrix} 1 & 2 & 3 \\ 505 & 606 & 707 \\ 1 & 2 & 3 \end{vmatrix}$ by using property

$= 100 \times 0$ (R_1 and R_3 are identical)

$= 0$

ii) $\begin{vmatrix} 312 & 313 & 314 \\ 315 & 316 & 317 \\ 318 & 319 & 320 \end{vmatrix} = 0$

L.H.S. $= \begin{vmatrix} 312 & 313 & 314 \\ 315 & 316 & 317 \\ 318 & 319 & 320 \end{vmatrix}$

$C_2 \rightarrow C_2 - C_1$

$= \begin{vmatrix} 312 & 1 & 314 \\ 315 & 1 & 317 \\ 318 & 1 & 320 \end{vmatrix}$

$C_3 \rightarrow C_3 - C_1$

$= \begin{vmatrix} 312 & 1 & 2 \\ 315 & 1 & 2 \\ 318 & 1 & 2 \end{vmatrix}$

take 2 common from C_3

$= 2 \begin{vmatrix} 312 & 1 & 1 \\ 315 & 1 & 1 \\ 318 & 1 & 1 \end{vmatrix}$

$= 2\,(0)$ (C_2 and C_3 are identical)

Ex. 2) Prove that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

L.H.S. $= \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

$R_1 \rightarrow aR_1$

$$= \frac{1}{a}\begin{vmatrix} a & a^2 & abc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

$R_2 \rightarrow bR_2$

$$= \frac{1}{a}\times\frac{1}{b}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ 1 & c & ab \end{vmatrix}$$

$R_3 \rightarrow cR_3$

$$= \frac{1}{a}\times\frac{1}{b}\times\frac{1}{c}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

$$= \frac{1}{abc}\times \text{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

(taking abc common from C_3)

$$= \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$C_1 \leftrightarrow C_3$

$$= (-1)\begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix}$$

$C_2 \leftrightarrow C_3$

$$= (-1)(-1)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \text{R.H.S.}$$

Ex. 3) If $\begin{vmatrix} x & y & z \\ -x & y & z \\ x & -y & z \end{vmatrix} = k.xyz$ then find the value of k

Sloution :

$$\text{L.H.S.} = \begin{vmatrix} x & y & z \\ -x & y & z \\ x & -y & z \end{vmatrix}$$

$R_2 \rightarrow R_2 + R_1$

$$= \begin{vmatrix} x & y & z \\ 0 & 2y & 2z \\ x & -y & z \end{vmatrix}$$

$R_3 \rightarrow R_3 + R_1$

$$= \begin{vmatrix} x & y & z \\ 0 & 2y & 2z \\ 2x & 0 & 2z \end{vmatrix}$$

$$= 2 \times 2\begin{vmatrix} x & y & z \\ 0 & y & z \\ x & 0 & z \end{vmatrix}$$ taking (2 common from R_2 and R_3)

$= 4[x(yz) - y(0 - xz) + z(0 - xy)]$

$= 4[xyz + xyz - xyz]$

$= 4xyz$

From given condition

L.H.S. = R.H.S.

4xyz = k xyz

$\therefore$ k = 4

EXERCISE 4.2

Q.1) Without expanding evaluate the following determinants.

i) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ ii) $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$ iii) $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$

Q.2) Prove that $\begin{vmatrix} x+y & y+z & z+x \\ z+x & x+y & y+z \\ y+z & z+x & x+y \end{vmatrix}$

$= 2\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}$

Q.3) Using properties of determinant show that

i) $\begin{vmatrix} a+b & a & b \\ a & a+c & c \\ b & c & b+c \end{vmatrix} = 4abc$

ii) $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} = 0$

Q.5) Solve the following equations.

i) $\begin{vmatrix} x+2 & x+6 & x-1 \\ x+6 & x-1 & x+2 \\ x-1 & x+2 & x+6 \end{vmatrix} = 0$

ii) $\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$

Q.6) If $\begin{vmatrix} 4+x & 4-x & 4-x \\ 4-x & 4+x & 4-x \\ 4-x & 4-x & 4+x \end{vmatrix} = 0$ then find the values of x

Q.7) Without expanding determinants show that

$$\begin{vmatrix} 1 & 3 & 6 \\ 6 & 1 & 4 \\ 3 & 7 & 12 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 & 3 \\ 2 & 1 & 2 \\ 1 & 7 & 6 \end{vmatrix} = 10\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 7 \\ 3 & 2 & 6 \end{vmatrix}$$

4.3 APPLICATIONS OF DETERMINANTS

4.3.1 Cramer's Rule

In linear algebra Cramer's rule is an explicit formula for the solution of a system of linear equations in many variables. In previous class we studied this with two variables. Our goal here is to expand the application of Cramer's rule to three equations in three variables (unknowns) . Variables are usually denoted by x, y and z.

Theorem - Consider the following three linear equations in variables three x, y, z.

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

Here a_i , b_i , c_i and d_i are constants.

The solution of this system of equations is

$$x = \frac{Dx}{D}, y = \frac{Dy}{D}, z = \frac{Dz}{D}$$

provided $D \neq 0$ where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad Dx = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$Dy = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad Dz = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Remark :

1) You will find the proof of the Cramer's Rule in QR code.

2) If D = 0 then there is no unique solution for the given system of equations.

SOLVED EXAMPLES

Ex. 1) Solve the following equation by using Cramer's rule.

$x+y+z = 6,\ \ x-y+z = 2,\ \ x+2y-z = 2$

Solution : Given equations are

$x+y+z = 6 \quad x-y+z = 2 \; x+2y-z = 2$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$= 1(1-2) - 1(-1-1) + 1(2+1)$

$= -1 + 2 + 3$

$= -1 + 5$

$= 4$

$$Dx = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ 2 & 2 & -1 \end{vmatrix}$$

$= 6(1-2) - 1(-2-2) + 1(4+2)$

$= -6 + 4 + 6$

$= 4$

$$Dy = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$= 1(-2-2) - 6(-1-1) + 1(2-2)$

$= -4 + 12 + 0$

$= 8$

$$Dz = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 1 & 2 & 2 \end{vmatrix}$$

$= 1(-2-4) - 1(2-2) + 6(2+1)$

$= -6 + 0 + 18$

$= 12$

$\therefore\ x = \dfrac{Dx}{D} = \dfrac{4}{4} = 1,\ y = \dfrac{Dy}{D} = \dfrac{8}{4} = 2$ and $z\ \dfrac{Dz}{D} = \dfrac{12}{4} = 3$ are solutions of given equation.

Ex. 2) By using Cramer's rule solve the following linear equations.

$x + y - z = 1,\ \ 8x + 3y - 6z = 1,\ -4x - y + 3z = 1$

Solution : Given equations are

$x + y - z = 1$

$8x + 3y - 6z = 1$

$-4x - y + 3z = 1$

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 8 & 3 & -6 \\ -4 & -1 & 3 \end{vmatrix}$$

$= 1(9-6) - 1(24-24) - 1(-8+12)$

$= 3 + 0 - 4$

$= -1$

$$Dx = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 3 & -6 \\ 1 & -1 & 3 \end{vmatrix}$$

$= 1(9-6) - 1(3+6) - 1\ (-1-3)$

$= 3 - 9 + 4$

$= -2$

$$Dy = \begin{vmatrix} 1 & 1 & -1 \\ 8 & 1 & -6 \\ -4 & 1 & 3 \end{vmatrix}$$

$= 1(3+6) - 1(24-24) - 1(8+4)$

$= 9 - 0 - 12$

$= -3$

$$Dz = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 3 & 1 \\ -4 & -1 & 1 \end{vmatrix}$$

$= 1(3+1) - 1(8+4) + 1(-8+12)$

$= 4 - 12 + 4$

$= 8 - 12$

$= -4$

$\therefore\ x = \dfrac{Dx}{D} = \dfrac{-2}{-1} = 2,\ y = \dfrac{Dy}{D} = \dfrac{-3}{-1} = 3$ and

$\therefore\ z = \dfrac{Dz}{D} = \dfrac{-4}{-1} = 4$

$\therefore x = 2,\ y = 3,\ z = 4$ are the solutions of the given equations.

Ex. 3) Solve the following equations by using determinant

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=-2\,,\quad \frac{1}{x}-\frac{2}{y}+\frac{1}{z}=3\,,\quad \frac{2}{x}-\frac{1}{y}+\frac{3}{2}=-1$$

Solution :

Put $\frac{1}{x}=p \quad \frac{1}{y}=q \quad \frac{1}{z}=r$

$\therefore$ Equations are

p + q + r = –2

p – 2q + r = 3

2p – q + 3r = –1

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 2 & -1 & 3 \end{vmatrix}$$

$= 1(-6+1) - 1(3-2) + 1(-1+4)$
$= -5 - 1 + 3$
$= -3$

$$Dp = \begin{vmatrix} -2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & -1 & 3 \end{vmatrix}$$

$= -2(-6+1) - 1(9+1) + 1(-3-2)$
$= 10 - 10 - 5$
$= -5$

$$Dq = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 3 & 1 \\ 2 & -1 & 3 \end{vmatrix}$$

$= 1(9+1) + 2(3\text{-}2) + 1(-1-6)$
$= 10 + 2 - 7$
$= 5$

$$Dr = \begin{vmatrix} 1 & 1 & -2 \\ 1 & -2 & 3 \\ 2 & -1 & -1 \end{vmatrix}$$

$= 1(2+3) - 1(-1-6) - 2(-1+4)$
$= 5 + 7 - 6$
$= 6$

$\therefore \quad p = \frac{Dp}{D} = \frac{-5}{-3} = \frac{5}{3}$,

$q = \frac{Dq}{q} = \frac{5}{-3} = \frac{-5}{3}$,

$r = \frac{Dr}{D} = \frac{6}{-3} = -2$

$\therefore \frac{1}{x} = p = \frac{5}{3} \quad \therefore x = \frac{3}{5}$,

$\therefore \frac{1}{y} = q = \frac{-5}{3} \quad \therefore y = \frac{-3}{5}$,

$\therefore \frac{1}{z} = r = -2 \quad \therefore z = \frac{-1}{2}$

$\therefore x = \frac{3}{5}$, $y = \frac{-3}{5}$, $z = \frac{-1}{2}$ are the solutions of the equations.

Ex. 4) The cost of 2 books, 6 notebooks and 3 pens is Rs.120. The cost of 3 books, 4 notebooks and 2 pens is Rs.105. while the cost of 5 books, 7 notebooks and 4 pens is Rs.183. Using this information find the cost of 1 book, 1 notebook and 1 pen.

Solution : Let Rs. x, Rs. y and Rs. z be the cost of one book, one notebook and one pen respectively. Then by given information we have,

$2x + 6y + 3z = 120$

$3x + 4y + 2z = 105$

$5x + 7y + 4z = 183$

$$D = \begin{vmatrix} 2 & 6 & 3 \\ 3 & 4 & 2 \\ 5 & 7 & 4 \end{vmatrix}$$

$= 2(16-14) - 6(12-10) + 3(21-20)$

$= 2(2) - 6(2) + 3(1)$

$= 4 - 12 + 3$

$= 7 - 12$

$= -5$

$$Dx = \begin{vmatrix} 120 & 6 & 3 \\ 105 & 4 & 2 \\ 183 & 7 & 4 \end{vmatrix} = 3\begin{vmatrix} 40 & 6 & 3 \\ 35 & 4 & 2 \\ 61 & 7 & 4 \end{vmatrix}$$

$= 3\,[40(16–14) – 6(140–122) + 3(245–244)]$

$= 3[40(2) – 6(18) + 3(1)]$

$= 3[80 –108 +3]$

$= 3[\ 83 – 108]$

$= 3[– 25] = –75$

$$Dy = 3\begin{vmatrix} 2 & 120 & 3 \\ 3 & 105 & 2 \\ 5 & 183 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 40 & 3 \\ 3 & 35 & 2 \\ 5 & 61 & 4 \end{vmatrix}$$

$= 3[2(140–122) – 40(12–10) + 3(183–175)]$

$= 3[\ 2(18) – 40(2) + 3(8)]$

$= 3[\ 36–80+24]$

$= 3[\ 60–80]$

$= 3[–20]$

$= –60$

$$Dz = 3\begin{vmatrix} 2 & 6 & 40 \\ 3 & 4 & 35 \\ 5 & 7 & 61 \end{vmatrix} = \begin{vmatrix} 2 & 6 & 120 \\ 3 & 4 & 105 \\ 5 & 7 & 183 \end{vmatrix}$$

$= 3[2(244–245) – 6(183–175) + 40(21–20)]$

$= 3[2(–1) – 6(8) + 40(1)]$

$= 3[–2 – 48 + 40\ ]$

$= 3[–50+40]$

$= 3[–10]$

$= –30$

$$\therefore\ x = \frac{Dx}{D} = \frac{-75}{-5} = 15,\ y = \frac{Dy}{D} = \frac{-60}{-5} = 12,$$

$$z = \frac{Dz}{D} = \frac{-30}{-5} = 6$$

∴ Rs.15, Rs. 12, Rs. 6 are the costs of one book, one notebook and one pen respectively.

4.3.2 Consistency of three equations in two variables

Consider the system of three linear equations in two variables x and y

$$\left.\begin{aligned} a_1x + b_1y + c_1 &= 0 \\ a_2x + b_2y + c_2 &= 0 \\ a_3x + b_3y + c_3 &= 0 \end{aligned}\right\} \quad \begin{aligned} &(I) \\ &(II) \\ &(III) \end{aligned}$$

These three equations are said to be consistent if they have a common solution.

Theorem : The necessary condition for the equation $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ to be consistent is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Proof : Consider the system of three linear equations in two variables x and y.

$$\left.\begin{aligned} a_1x + b_1y + c_1 &= 0 \\ a_2x + b_2y + c_2 &= 0 \\ a_3x + b_3y + c_3 &= 0 \end{aligned}\right\} \quad (I)$$

We shall now obtain the necessary condition for the system (I) be consistent.

Consider the solution of the equations

$$a_2x+b_2y = -\,c_2$$
$$a_3x+b_3y = -\,c_3$$

If $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \neq 0$ then by Cramer's Rule the system of two unknowns, we have

$$x = \frac{\begin{vmatrix} -c_2 & b_2 \\ -c_3 & b_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}\ ,\quad y = \frac{\begin{vmatrix} a_2 & -c_2 \\ a_3 & -c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}$$ put these

values in equation $a_1x + b_1y + c_1 = 0$ then

$$a_1 \frac{\begin{vmatrix} -c_2 & b_2 \\ -c_3 & b_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} + b_1 \frac{\begin{vmatrix} a_2 & -c_2 \\ a_3 & -c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} + c_1 = 0$$

i.e $a_1 \begin{vmatrix} -c_2 & b_2 \\ -c_3 & b_3 \end{vmatrix} + b_1 \begin{vmatrix} a_2 & -c_2 \\ a_3 & -c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$

i.e $-a_1 \begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$

i.e $a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$

i.e $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Note : This is a necessary condition that the equations are consistant. The above condition of consistency in general is not sufficient.

SOLVED EXAMPLES

Ex. 1) Verify the consistency of following equations

$2x+2y = -2$, $x + y = -1$, $3x + 3y = -5$

Solution : By condition of consistency consider

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 3 & 3 & 5 \end{vmatrix}$$

$= 2(5-3) - 2(5-3) + 2(3-3) = 4 - 4 + 0 = 0$

But the equations have no common Solution. (why?)

Ex. 2) Examine the consistency of following equations.

i) $x + y = 2$, $2x + 3y = 5$, $3x - 2y = 1$

Solution : Write the given equation in standard form.

$x + y - 2 = 0$, $2x + 3y - 5 = 0$, $3x - 2y - 1 = 0$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 3 & -5 \\ 3 & -2 & -1 \end{vmatrix}$$

$= 1(-3-10) - 1(-2+15) - 2(-4-9)$

$= -13 - 13 + 26 = 0$

$\therefore$ Given equations are consistent.

ii) $x + 2y - 3 = 0$, $7x + 4y - 11 = 0$, $2x + 3y + 1 = 0$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -3 \\ 7 & 4 & -11 \\ 2 & 3 & 1 \end{vmatrix}$$

$= 1(4+33) - 2(7+22) - 3(21-8)$

$= 37 - 58\ 39 \qquad = 37 - 97$

$= -60 \neq 0$

$\therefore$ Given system of equations is not consistent.

iii) $x + y = 1$, $2x + 2y = 2$, $3x + 3y = 5$

Solution : $x + y = 1$, $2x + 2y = 2$, $3x + 3y = 5$ are given equations.

Check the condition of consistency.

$$\begin{vmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 3 & 3 & -5 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 3 & 3 & -5 \end{vmatrix}$$

$= 2(0) = 0$ (R_1 and R_2 are identical)

Let us examine further.

Note that lines given by the equations $x + y = 1$ and $3x + 3y = 5$ are parallel to each other. They do not have a common solution, so equations are not consistent.

Ex. 3) Find the value of k if the following equations are consistent.

$7x - ky = 4$, $2x + 5y = 9$ and $2x + y = 8$

Solution : Given equations are

$7x - ky - 4 = 0$, $2x + 5y - 9 = 0$,

$2x + y - 8 = 0$ are consistent

$$\therefore \begin{vmatrix} 7 & -k & 4 \\ 2 & 5 & -9 \\ 2 & 1 & -8 \end{vmatrix} = 0$$

$\therefore$ 7(–40+9) + k(–16+18) –4(2–10) = 0

$\therefore$ 7(–31) +2k –4(–8) = 0

$\therefore$ –217 + 2k + 32 = 0 –185 + 2k = 0

$\therefore$ 2k = 185 $\quad \therefore$ k = $\frac{185}{2}$

4.3.3 Area of triangle and Collinearity of three points.

Theorem : If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of triangle ABC then the area of triangle is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Proof : Consider a triangle ABC in Cartesian coordinate system. Draw AP, CQ and BR perpendicular to the X axis

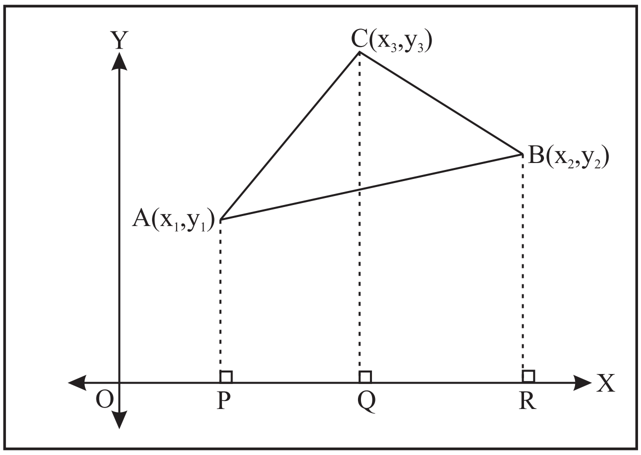


Fig. 4.1

From the figure,

Area of ΔABC = Area of trapezium PACQ +Area of trapezium QCBR –Area of trapezium PABR

Area of triangle ABC = $\frac{1}{2}$ PQ.[AP+CQ]

$+ \frac{1}{2}$ QR.[QC+BR]– $\frac{1}{2}$ PR.[AP+BR]

$= \frac{1}{2}(y_1+y_3)(x_3-x_1) + \frac{1}{2}(y_2+y_3)(x_2-x_3)$

$- \frac{1}{2}(y_1+y_2)(x_2-x_1)$

$= \frac{1}{2}[y_1.x_3 - x_1y_1 + x_3y_3 - x_1y_3 + x_2y_2 - x_3y_2 + x_2y_3$
$- x_3y_3 - x_2y_1 + x_1y_1 - x_2y_2 + x_1y_1]$

$= \frac{1}{2}[y_1x_3 - x_1y_3 - x_3y_2 + x_2y_3 - x_2y_1 + x_1y_2]$

$= \frac{1}{2}[x_1(y_2-y_3) - x_2(y_1-y_3) + x_3(y_1-y_2)]$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Remark:

i) Area is a positive quantity. Hence we always take the absolute value of a determinant.

ii) If area is given, consider both positive and negative values of the determinant for calculation of unknown co-ordinates.

iii) If area of a triangle is zero then the given three points are **collinear**.

SOLVED EXAMPLES

Ex. 1) Find the area of the triangle whose vertices are A(–2, –3) , B(3, 2) and C(–1, –8)

Solution : Given $(x_1, y_1) = (-2, -3)$, $(x_2, y_2) = (3, 2)$, and $(x_3, y_3) = (-1, -8)$

We know that area of triangle

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$$

$= \frac{1}{2}[-2(2+8)+3(3+1)+1(-24+2)]$

$= \frac{1}{2}[-20+12-22]$

$= \frac{1}{2}[-42+12] \quad = \frac{1}{2}[-30] \quad = -15$

Area is positive.

$\therefore$ Area of triangle = 15 square unit

This gives the area of the triangle ABC in that order of the vertices. If we consider the same triangle as ACB, then triangle is considered in opposite orientation. The area then is 15 sq. units. This also agrees with the rule that interchanging 2nd and 3rd rows changes the sign of the determinant.

Ex. 2) If the area of triangle with vertices P(–3, 0), Q(3, 0) and R(0, K) is 9 square unit then find the value of k.

Solution : Given $(x_1, y_1) \equiv (-3, 0)$, $(x_2, y_2) \equiv (3, 0)$ and $(x_3, y_3) \equiv (0, k)$ and area of Δ is 9 sq. unit.

We know that area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$\therefore \pm 9 = \frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix}$ (Area is positive but the determinant can be of either sign)

$\therefore \pm 9 = \frac{1}{2}[-3(0-k)+1(3k-0)]$

$\therefore \pm 9 = \frac{1}{2}[3+3k] \quad \therefore \pm 9 = 3k \quad \therefore k = \pm 3$

Ex. 3) Find the area of triangle whose vertices are A(3, 7) B(4, –3) and C(5, –13). Interpret your answer.

Solution : Given $(x_1, y_1) \equiv (3, 7)$, $(x_2, y_2) \equiv (4, -3)$ and $(x_3, y_3) \equiv (5, -13)$

Area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 3 & 7 & 1 \\ 4 & -3 & 1 \\ 5 & -13 & 1 \end{vmatrix}$

$= \frac{1}{2}[3(-3+13) - 7(4-5) + 1(-52+15)]$

$= \frac{1}{2}[30+7-37] \quad = \frac{1}{2}[37-37] = 0$

$A(\Delta ABC) = 0 \quad \therefore$ A, B, C are collinear points

Ex. 4) Show that the following points are collinear by determinant method.

A(2, 5), B(5, 7), C(8, 9)

Solution : Given $A \equiv (x_1, y_1) = (2, 5)$, $B \equiv (x_2, y_2) \equiv (5, 7)$, $C \equiv (x_3, y_3) \equiv (8,9)$

If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ then, A, B, C are collinear

$\therefore \begin{vmatrix} 2 & 5 & 1 \\ 5 & 7 & 1 \\ 8 & 9 & 1 \end{vmatrix} = 2(7-9) - 5(5-8) + 1(45-56)$

$= -4 + 15 - 11 \quad = -15 + 15 = 0$

$\therefore$ A, B, C are collinear.

EXERCISE 4.3

Q.1) Solve the following linear equations by using Cramer's Rule.

i) $x+y+z = 6$, $x-y+z = 2$, $x+2y-z = 2$

ii) $x+y-2z = -10$,
$2x+y-3z = -19$, $4x+6y+z = 2$

iii) $x+z = 1$, $y+z = 1$, $x+y = 4$

iv) $\frac{-2}{x}-\frac{1}{y}-\frac{3}{z}=3$, $\frac{2}{x}-\frac{3}{y}+\frac{1}{z}=-13$

and $\frac{2}{x}-\frac{3}{z}=-11$

Q.2) The sum of three numbers is 15. If the second number is subtracted from the sum of first and third numbers then we get 5. When the third number is subtracted from the sum of twice the first number and the second number, we get 4. Find the three numbers.

Q.3) Examine the consistency of the following equations.

i) $2x-y+3=0$, $3x+y-2=0$, $11x+2y-3=0$

ii) $2x+3y-4=0$, $x+2y=3$, $3x+4y+5=0$

iii) $x+2y-3=0$, $7x+4y-11=0$, $2x+4y-6=0$

Q.4) Find k if the following equations are consistent.

i) $2x+3y-2=0$, $2x+4y-k=0$, $x-2y+3k=0$

ii) $kx+3y+1=0$, $x+2y+1=0$, $x+y=0$

Q.5) Find the area of triangle whose vertices are

i) A(5,8), B(5,0) C(1,0)

ii) P($\frac{3}{2}$, 1), Q(4, 2), R(4, $\frac{-1}{2}$)

iii) M(0, 5), N(–2, 3), T(1, –4)

Q.6) Find the area of quadrilateral whose vertices are

A(–3, 1), B(–2, –2), C(3,–1), D(1,4)

Q.7) Find the value of k, if the area of triangle whose vertices are P(k, 0), Q(2, 2), R(4, 3) is $\frac{3}{2}$ sq.unit

Q.8) Examine the collinearity of the following set of points

i) A(3, –1), B(0, –3), C(12, 5)

ii) P(3, –5), Q(6, 1), R(4, 2)

iii) L(0, $\frac{1}{2}$), M (2, –1), N(–4, $\frac{7}{2}$)

MISCELLANEOUS EXERCISE - 4 (A)

(I) Select the correct option from the given alternatives.

Q.1 The determinant D $=\begin{vmatrix} a & b & a+b \\ b & c & b+c \\ a+b & b+c & 0 \end{vmatrix}=0$ If

A) a,b, c are in A.P.

B) a , b, c are in G.P.

C) a, b, c are in H.P.

D) α is root of $ax^2+2bx+c=0$

Q.2 If $\begin{vmatrix} x^k & x^{k+2} & x^{k+3} \\ y^k & y^{k+2} & y^{k+3} \\ z^k & z^{k+2} & z^{k+3} \end{vmatrix}=(x-y)(y-z)(z-x)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$ then

A) k= –3 B) k = –1 C) k = 1 D) k = 3

Q.3 Let $D=\begin{vmatrix} \sin\theta.\cos\phi & \sin\theta.\sin\phi & \cos\theta \\ \cos\theta.\cos\phi & \cos\theta.\sin\phi & -\sin\theta \\ -\sin\theta.\sin\phi & \sin\theta.\cos\phi & 0 \end{vmatrix}$ then

A) D is independent of θ

B) D is independent of ϕ

C) D is a constant

D) $\frac{dD}{d}$ at θ =π/2 is equal to 0

Q.4 The value of a for which system of equation $a^3x+(a+1)^3y+(a+2)^3z=0$ $ax+(a+1)y+(a+2)z=0$ and $x+y+z=0$ has non zero Soln. is

A) 0 B) –1 C) 1 D) 2

Q.5 $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} =$

A) $2\begin{vmatrix} c & b & a \\ r & q & p \\ z & y & x \end{vmatrix}$ B) $2\begin{vmatrix} b & a & c \\ q & p & r \\ y & x & z \end{vmatrix}$ C) $2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

D) $2\begin{vmatrix} a & c & b \\ p & r & q \\ x & z & y \end{vmatrix}$

Q.6 The system $3x - y + 4z = 3$, $x + 2y - 3z = -2$ and $6x + 5y + \lambda z = -3$ has at least one Solution when

A) $\lambda = -5$ B) $\lambda = 5$

C) $\lambda = 3$ D) $\lambda = -13$

Q.7 If $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$ has other two roots are

A) 2, –7 B) –2, 7 C) 2, 7 D) –2, –7

Q.8 If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$ then

A) $x = 3, y = 1$ B) $x = 1, y = 3$

C) $x = 0, y = 3$ D) $x = 0, y = 0$

Q.9 If A(0,0), B(1,3) and C(k,0) are vertices of triangle ABC whose area is 3 sq.units then value of k is

A) 2 B) –3 C) 3 or –3 D) –2 or +2

Q.10 Which of the following is correct

A) Determinant is square matrix

B) Determinant is number associated to matrix

C) Determinant is number associated to square matrix

D) None of these

(II) Answer the following questions.

Q.1) Evaluate i) $\begin{vmatrix} 2 & -5 & 7 \\ 5 & 2 & 1 \\ 9 & 0 & 2 \end{vmatrix}$ ii) $\begin{vmatrix} 1 & -3 & 12 \\ 0 & 2 & -4 \\ 9 & 7 & 2 \end{vmatrix}$

Q.2) Evaluate determinant along second column

$\begin{vmatrix} 1 & -1 & 2 \\ 3 & 2 & -2 \\ 0 & 1 & -2 \end{vmatrix}$

Q.3) Evaluate i) $\begin{vmatrix} 2 & 3 & 5 \\ 400 & 600 & 1000 \\ 48 & 47 & 18 \end{vmatrix}$

ii) $\begin{vmatrix} 101 & 102 & 103 \\ 106 & 107 & 108 \\ 1 & 2 & 3 \end{vmatrix}$ by using properties

Q.4) Find minor and cofactor of elements of the determinant.

i) $\begin{vmatrix} -1 & 0 & 4 \\ -2 & 1 & 3 \\ 0 & -4 & 2 \end{vmatrix}$ ii) $\begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{vmatrix}$

Q.5) Find the value of x if

i) $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & -5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ ii) $\begin{vmatrix} 1 & 2x & 4x \\ 1 & 4 & 16 \\ 1 & 1 & 1 \end{vmatrix} = 0$

Q.6) By using properties of determinant prove that $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$

Q.7) Without expanding determinant show that

i) $\begin{vmatrix} b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \\ a+b & ab & a^2b^2 \end{vmatrix} = 0$ ii) $\begin{vmatrix} xa & yb & zc \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} x & y & z \\ a & b & c \\ bc & ca & ab \end{vmatrix}$

iii) $\begin{vmatrix} l & m & n \\ e & d & f \\ u & v & w \end{vmatrix} = \begin{vmatrix} n & f & w \\ l & e & u \\ m & d & v \end{vmatrix}$

iv) $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$

Q.8) If $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$ =0 then show that

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=1$$

Q.9) Solve the following linear equations by Cramer's Rule.

i) $2x-y+z = 1$, $x+2y+3z = 8$, $3x+y-4z =1$

ii) $\frac{1}{x}+\frac{1}{y}=\frac{3}{2}$, $\frac{1}{y}+\frac{1}{z}=\frac{5}{6}$, $\frac{1}{z}+\frac{1}{x}=\frac{4}{3}$

iii) $2x+3y+3z=5$, $x-2y+z = -4$,
$3x-y-2z=3$

iv) $x-y+2z=7$, $3x+4y-5z=5$, $2x-y+3z=12$

Q.10) Find the value of k if the following equation are consistent.

i) $(k+1)x+(k-1)y+(k-1) = 0$

$(k-1)x+(k+1)y+(k-1) = 0$

$(k-1)x+(k-1)y+(k+1) = 0$

ii) $3x+y-2=0$ $kx+2y-3=0$ and $2x-y = 3$

iii) $(k-2)x +(k-1)y =17$,
$(k-1)x+ (k-2)y = 18$ and $x + y = 5$

Q.11) Find the area of triangle whose vertices are

i) A(–1,2), B(2,4), C(0,0)

ii) P(3,6), Q(–1,3), R (2,–1)

iii) L(1,1), M (–2,2), N (5,4)

Q.12) Find the value of k

i) If area of triangle is 4 square unit and vertices are P(k, 0), Q(4, 0), R(0, 2)

ii) If area of triangle is 33/2 square unit and vertices are L (3,–5), M(–2,k), N (1,4)

Q.13) Find the area of quadrilateral whose vertices are A (0, –4), B(4, 0) , C(–4, 0), D (0, 4)

Q.14) An amount of ₹ 5000 is put into three investments at the rate of interest of 6% , 7% and 8% per annum respectively. The total annual income is ₹ 350. If the combined income from the first two investments is ₹ 70 more than the income from the third. Find the amount of each investment.

Q.15) Show that the lines $x-y=6$, $4x-3y=20$ and $6x+5y+8=0$ are concurrent .Also find the point of concurrence

Q.16) Show that the following points are collinear by determinant

a) L (2,5), M(5,7), N(8,9)

b) P(5,1), Q(1,–1), R(11,4)

Further Use of Determinants

1) To find the volume of parallelepiped and tetrahedron by vector method

2) To state the condition for the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ representing a pair of straight lines.

3) To find the shortest distance between two skew lines.

4) Test for intersection of two line in three dimensional geometry.

5) To find cross product of two vectors and scalar triple product of vectors

6) Formation of differential equation by eliminating arbitrary constant.

Let's Learn

4.4 Introduction to Matrices :

The theory of matrices was developed by a Mathematician Arthur Cayley. Matrices are useful in expressing numerical information in compact form. They are effectively used in expressing different operators. Hence in Economics, Statistics and Computer science they are essential.

Definition : A rectangular arrangement of *mn* numbers in m rows and n columns, enclosed in [ ] or ( ) is called a matrix of order *m* by *n*.

A matrix by itself does not have a value or any special meaning.

Order of the matrix is denoted by $m \times n$, read as *m* by *n*.

Each member of the matrix is called an element of the matrix.

Matrices are generally denoted by A, B, C ,… and their elements are denoted by a_{ij}, b_{ij}, c_{ij}, … etc. e.g. a_{ij} is the element in *i*th row and *j*th column of the matrix.

For example, i) $A = \begin{bmatrix} 2 & -3 & 9 \\ 1 & 0 & -7 \\ 4 & -2 & 1 \end{bmatrix}$ Here $a_{32} = -2$

A is a matrix having 3 rows and 3 columns. The order of A is 3×3, read as three by three. There are 9 elements in matrix A.

ii) $B = \begin{bmatrix} -1 & -5 \\ 2 & 0 \\ 6 & 9 \end{bmatrix}$

B is a matrix having 3 rows and 2 columns. The order of B is 3×2. There are 6 elements in matrix B.

iii) $C = \begin{bmatrix} 1+i & 8 \\ i & -3i \end{bmatrix}$, C is a matrix of order 2×2.

iv) $D = \begin{bmatrix} -1 & 9 & 2 \\ 3 & 0 & -3 \end{bmatrix}$, D is a matrix of order 2×3.

In general a matrix of order m x n is represented by

$$A = [a_{ij}]_{mxn} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3j} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Here a_{ij} = An element in i^{th} row and j^{th} column.

Ex. In matrix $A = \begin{bmatrix} 2 & -3 & 9 \\ 1 & 0 & -7 \\ 4 & -2 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$a_{11} = 2$, $a_{12} = -3$, $a_{13} = 9$, $a_{21} = 1$, $a_{22} = 0$, $a_{23} = -7$, $a_{31} = 4$, $a_{32} = -2$, $a_{33} = 1$

4.4.1 Types of Matrices :

1) **Row Matrix :** A matrix having only one row is called as a row matrix. It is of order 1 x n, Where $n \geq 1$.

Ex. i) $[-1 \quad 2]_{1\times2}$ ii) $[0 \quad -3 \quad 5]_{1\times3}$

2) **Column Matrix :** A matrix having only one column is called as a column matrix. It is of order m x 1, Where $m \geq 1$.

Ex. i) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2x1}$ ii) $\begin{bmatrix} 5 \\ -9 \\ -3 \end{bmatrix}_{3x1}$

Note : Single element matrix is row matrix as well as column matrix. e.g. $[5]_{1x1}$

3) Zero or Null matrix : A matrix in which every element is zero is called as a zero or null matrix. It is denoted by O.

Ex. i) $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3x3}$

ii) $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3x2}$

4) Square Matrix : A matrix with equal number of rows and coloumns is called a square matrix.

Examples, i) $A = \begin{bmatrix} 5 & -3 & i \\ 1 & 0 & -7 \\ 2i & -8 & 9 \end{bmatrix}_{3x3}$

ii) $C = \begin{bmatrix} -1 & 0 \\ 1 & -5 \end{bmatrix}_{2x2}$

Note : A matrix of order n×n is also called as square matrix of order n.

Let A $= [a_{ij}]_{nxn}$ be a square matrix of order n then

(i) The elements $a_{11,}\ a_{22,}\ a_{33,\ldots}\ a_{ii\ \ldots}\ a_{nn}$ are called the diagonal elements of matrix A.

Note that the diagonal elements are defined only for a square matrix.

(ii) Elements $a_{ij,}$ where $i \neq j$ are called non diagonal elements of matrix A.

(iii) Elements $a_{ij,}$ where $i < j$ represent elements above the diagonal.

(iv) Elements $a_{ij,}$ where $i > j$ represent elements below the diagonal.

Statements iii) and iv) can be verified by observing square matrices of different orders.

5) Diagonal Matrix : A square matrix in which every non-diagonal element is zero, is called a diagonal matrix.

Ex. i) $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}_{3x3}$ = diag (5,0,9)

ii) $B = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}_{2x2}$

iii) $C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}_{2x2}$

Note: If $a_{11,}\ a_{22,}\ a_{33}$ are diagonal elements of a diagonal matrix A of order 3, then we write the matrix A as A = Diag.

6) Scalar Matrix : A diagonal matrix in which all the diagonal elements are same, is called as a scalar matrix.

For Ex. i) $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}_{3x3}$

ii) $B = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}_{2x2}$

7) Unit or Identity Matrix : A scalar matrix in which all the diagonal elements are 1(unity), is called a Unit Matrix or an Identity Matrix. An Identity matrix of order n is denoted by $I_{n.}$

Ex. i) $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ii) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Note:

1. Every Identity matrix is a Scalar matrix but every scalar matrix need not be Identity matrix. However a scalar matrix is a scalar multiple of the identity matrix.

2. Every scalar matrix is a diagonal matrix but every diagonal matrix need not be a scalar matrix.

8) Upper Triangular Matrix : A square matrix in which every element below the diagonal is zero, is called an upper triangular matrix. Matrix A $= [a_{ij}]_{nxn}$ is upper triangular if $a_{ij} = 0$ for all $i > j$.

For Ex. i) $A = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 9 \end{bmatrix}_{3x3}$

ii) $B = \begin{bmatrix} -3 & 1 \\ 0 & 8 \end{bmatrix}_{2x2}$

9) Lower Triangular Matrix : A square matrix in which every element above the diagonal is zero, is called a lower triangular matrix.

Matrix A $= [a_{ij}]_{nxn}$ is lower triangular if $a_{ij} = 0$ for all $i < j$.

For Ex. i) $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -5 & 1 & 9 \end{bmatrix}_{3x3}$

ii) $B = \begin{bmatrix} 7 & 0 \\ -1 & 3 \end{bmatrix}_{2x2}$

10) Triangular Matrix : A square matrix is called a triangular matrix if it is an upper triangular or a lower triangular matrix.

Note : The diagonal, scalar, unit and null matrices are also triangular matrices.

11) Symmetric Matrix : A square matrix A $= [a_{ij}]_{nxn}$ in which $a_{ij} = a_{ji,}$ for all i and j, is called a symmetric matrix.

Ex. i) $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}_{3x3}$

ii) $B = \begin{bmatrix} -3 & 1 \\ 1 & 8 \end{bmatrix}_{2x2}$

iii) $C = \begin{bmatrix} 2 & 4 & -7 \\ 4 & 5 & -1 \\ -7 & -1 & -3 \end{bmatrix}_{3x3}$

Note:

The scalar matrices are symmetric. A null square matrix is symmetric.

12) Skew-Symmetric Matrix : A square matrix A $= [a_{ij}]_{nxn}$ in which $a_{ij} = -a_{ji,}$ for all i and j, is called a skew symmetric matrix.

Here for $i = j$, $a_{ii} = -a_{ii,}$ $2a_{ii} = 0$ $a_{ii} = 0$ for all $i = 1, 2, 3, \ldots\ldots n$.

In a skew symmetric matrix each diagonal element is zero.

e.g. i) $A = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}_{2x2}$

ii) $B = \begin{bmatrix} 0 & 4 & -7 \\ -4 & 0 & 5 \\ 7 & -5 & 0 \end{bmatrix}_{3x3}$

Note : A null square matrix is also a skew symmetric.

13) Determinant of a Matrix : Determinant of a matrix is defined only for a square matrix.

If A is a square matrix, then the same arrangement of the elements of A also gives us a determinant, by replacing square brackets by vertical bars. It is denoted by |A| or det(A).

If A $= [a_{ij}]_{nxn,}$ then is of order n.

Ex. i) If $A = \begin{bmatrix} 1 & 3 \\ -5 & 4 \end{bmatrix}_{2x2}$

then $|A| = \begin{vmatrix} 1 & 3 \\ -5 & 4 \end{vmatrix}$

ii) If $B = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 1 & 5 \\ 7 & -5 & 0 \end{bmatrix}_{3x3}$

then $|B| = \begin{vmatrix} 2 & -1 & 3 \\ -4 & 1 & 5 \\ 7 & -5 & 0 \end{vmatrix}$

14) Singular Matrix : A square matrix A is said to be a singular matrix if $|A| = \det(A) = 0$, otherwise it is said to be a non-singular matrix.

Ex. i) If $A = \begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}_{2x2}$

then $|A| = \begin{vmatrix} 6 & 3 \\ 8 & 4 \end{vmatrix} = 24 - 24 = 0.$

Therefore A is a singular matrix.

ii) If $B = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}_{3x3}$ then

$|B| = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{vmatrix}$

$|B| = 2(24-25) - 3(18-20) + 4(15-16)$

$= -2 + 6 - 4 = 0$

$|B| = 0$ Therefore B is a singular matrix.

iii) $A = \begin{bmatrix} 2 & -1 & 3 \\ -7 & 4 & 5 \\ -2 & 1 & 6 \end{bmatrix}_{3X3}$ then

$|A| = \begin{vmatrix} 2 & -1 & 3 \\ -7 & 4 & 5 \\ -2 & 1 & 6 \end{vmatrix}$

$|A| = 2(24-5) - (-1)(-42+10) + 3(-7+8)$

$= 38 - 32 + 3 = 9$

$|A| = 9$, As $|A| \neq 0$, A is a non-singular matrix.

15) Transpose of a Matrix : The matrix obtained by interchanging rows and columns of matrix A is called Transpose of matrix A. It is denoted by A' or A^T. If A is matrix of order $m \times n$, then order of A^T is $n \times m$.

If $A^T = A' = B$ then $b_{ij} = a_{ji}$

e.g. i) If $A = \begin{bmatrix} -1 & 5 \\ 3 & -2 \\ 4 & 7 \end{bmatrix}_{3\times 2}$

then $A^T = \begin{bmatrix} -1 & 3 & 4 \\ 5 & -2 & 7 \end{bmatrix}_{2\times 3}$

ii) If $B = \begin{bmatrix} 1 & 0 & -2 \\ 8 & -1 & 2 \\ 4 & 3 & 5 \end{bmatrix}_{3\times 3}$

then $B^T = \begin{bmatrix} 1 & 8 & 4 \\ 0 & -1 & 3 \\ -2 & 2 & 5 \end{bmatrix}_{3\times 3}$

Remark:

1) If A is symmetric then $A = A^T$

2) If B is skew symmetric, then is $B = -B^T$

Activity :

Construct a matrix of order 2×2 where the a_{ij} th element is given by $a_{ij} = \dfrac{(i+j)^2}{2+i}$

Solution : Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2X2}$ be the required matrix.

Given that $a_{ij} = \dfrac{(i+j)^2}{2+i}$, $a_{11} = \dfrac{(\dots\dots)^2}{\dots\dots+1} = \dfrac{4}{3}$,

$a_{12} = \dfrac{(\dots\dots)^2}{\dots\dots\dots} = \dfrac{9}{3} = \dots\dots$

$a_{21} = \dfrac{(2+1)^2}{2+2} = \dfrac{\dots\dots}{4}$,

$$a_{22} = \frac{(\ldots\ldots\ldots)^2}{2+2} = \frac{\ldots\ldots\ldots}{\ldots\ldots\ldots} = 4$$

$$\therefore\ A = \begin{bmatrix} \frac{4}{3} & \cdots \\ \frac{\cdots\cdots}{\cdots\cdots} & 4 \end{bmatrix}$$

SOLVED EXAMPLES

Ex. 1) Show that the matrix $\begin{bmatrix} x+y & y+z & z+x \\ 1 & 1 & 1 \\ z & x & y \end{bmatrix}$ is a singular matrix.

Solution : Let $A = \begin{bmatrix} x+y & y+z & z+x \\ 1 & 1 & 1 \\ z & x & y \end{bmatrix}$

$$\therefore\ |A| = \begin{vmatrix} x+y & y+z & z+x \\ 1 & 1 & 1 \\ z & x & y \end{vmatrix}$$

Now $|A| = (x+y)(y-x)-(y+z)(y-z)+(z+x)(x-z)$

$= y^2 - x^2 - y^2 + z^2 + x^2 - z^2$

$= 0$

$\therefore$ A is a singular matrix.

Ex. 2) If $A = \begin{bmatrix} -1 & -5 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}_{3X2}$ Find $(A^T)^T$.

Solution : Let $A = \begin{bmatrix} -1 & -5 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}_{3X2}$

$$\therefore\ A^T = \begin{bmatrix} -1 & 2 & 3 \\ -5 & 0 & -4 \end{bmatrix}_{2x3}$$

$$\therefore\ (A^T)^T = \begin{bmatrix} -1 & -5 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}_{3X2} = A$$

Ex. 3) Find a, b, c if the matrix $A = \begin{bmatrix} 2 & a & 3 \\ -7 & 4 & 5 \\ c & b & 6 \end{bmatrix}$ is a symmetric matrix.

Solution : Given that $A = \begin{bmatrix} 2 & a & 3 \\ -7 & 4 & 5 \\ c & b & 6 \end{bmatrix}$ is a symmetric matrix.

$a_{ij} = a_{ji}$ for all i and j

As $a_{12} = a_{21}$

$\therefore$ a = –7

As $a_{32} = a_{23}$

$\therefore$ b = 5

As $a_{31} = a_{13}$

$\therefore$ c = 3

EXERCISE 4.4

(1) Construct a matrix $A = [a_{ij}]_{3\times2}$ whose elements a_{ij} are given by (i) $a_{ij} = \frac{(i-j)^2}{5-i}$

(ii) $a_{ij} = i - 3j$ (iii) $a_{ij} = \frac{(i+j)^3}{5}$

(2) Classify the following matrices as, a row, a column, a square, a diagonal, a scalar, a unit, an upper triangular, a lower triangular, a symmetric or a skew-symmetric matrix.

(i) $\begin{bmatrix} 3 & -2 & 4 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 4 & 7 \\ -4 & 0 & -3 \\ -7 & 3 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 5 \\ 4 \\ -3 \end{bmatrix}$ (iv) $\begin{bmatrix} 9 & \sqrt{2} & -3 \end{bmatrix}$

(v) $\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ -7 & 3 & 1 \end{bmatrix}$

(vii) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ (viii) $\begin{bmatrix} 10 & -15 & 27 \\ -15 & 0 & \sqrt{34} \\ 27 & \sqrt{34} & \frac{5}{3} \end{bmatrix}$

(ix) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (x) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(3) Which of the following matrices are singular or non singular ?

(i) $\begin{bmatrix} a & b & c \\ p & q & r \\ 2a-p & 2b-q & 2c-r \end{bmatrix}$

(ii) $\begin{bmatrix} 5 & 0 & 5 \\ 1 & 99 & 100 \\ 6 & 99 & 105 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 5 & 7 \\ -2 & 1 & 4 \\ 3 & 2 & 5 \end{bmatrix}$

(iv) $\begin{bmatrix} 7 & 5 \\ -4 & 7 \end{bmatrix}$

(4) Find k if the following matrices are singular

(i) $\begin{bmatrix} 7 & 3 \\ -2 & k \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 3 & 1 \\ 7 & k & 1 \\ 10 & 9 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} k-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$

(5) If $A = \begin{bmatrix} 5 & 1 & -1 \\ 3 & 2 & 0 \end{bmatrix}$, Find $(A^T)^T$.

(6) If $A = \begin{bmatrix} 7 & 3 & 1 \\ -2 & -4 & 1 \\ 5 & 9 & 1 \end{bmatrix}$, Find $(A^T)^T$.

(7) Find a, b, c if $\begin{bmatrix} 1 & \frac{3}{5} & a \\ b & -5 & -7 \\ -4 & c & 0 \end{bmatrix}$ is a symmetric matrix.

(8) Find x, y, z if $\begin{bmatrix} 0 & -5i & x \\ y & 0 & z \\ \frac{3}{2} & -\sqrt{2} & 0 \end{bmatrix}$ is a skew symmetric matrix.

(9) For each of the following matrices, using its transpose state whether it is a symmetric, a skew-symmetric or neither.

(i) $\begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ -5 & 4 & 9 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 5 & 1 \\ -5 & 4 & 6 \\ -1 & -6 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 1+2i & i-2 \\ -1-2i & 0 & -7 \\ 2-i & 7 & 0 \end{bmatrix}$

(10) Construct the matrix $A = [a_{ij}]_{3\times3}$ where $a_{ij} = i-j$. State whether A is symmetric or skew symmetric.

4.5 Algebra of Matrices :

(1) Equality of matrices

(2) Multiplication of a matrix by a scalar

(3) Addition of matrices

(4) Multiplication of two matrices.

(1) Equality of matrices : Two matrices A and B are said to be equal if (i) order of A = order of B and (ii) corresponding elements of A and B are same, that is if $a_{ij}=b_{ij}$ for; all i,j and symbolically written as A=B.

Ex. (i) If $A \begin{bmatrix} 15 & 14 \\ 12 & 10 \end{bmatrix}_{2x2}$

$B = \begin{bmatrix} 15 & 14 \\ 10 & 12 \end{bmatrix}_{2x2}$ and $C = \begin{bmatrix} 15 & 14 \\ 10 & 12 \end{bmatrix}_{2x2}$

Here $A \neq B$, $A \neq C$ but $B = C$ by definition of equality.

Ex. (ii) If $\begin{bmatrix} 2a-b & 4 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -7 & a+3b \end{bmatrix}$

then using definition of equality of matrices, we have $2a-b=1$ ……(1) and $a+3b=2$ …… (2)

Solving equations (1) and (2), we get $a = \frac{5}{7}$ and $b = \frac{3}{7}$

(2) Multiplication of a Matrix by a scalar: If A is any matrix and k is a scalar, then the matrix obtained by multiplying each element of A by the scalar k is called the scalar multiple of the given matrix A and is denoted by kA.

Thus if $A = [a_{ij}]_{mxn}$ and k is any scalar then $kA = [ka_{ij}]_{mxn}$.

Here the order of matrix A and kA are same.

Ex. $A = \begin{bmatrix} -1 & 5 \\ 3 & -2 \\ 4 & 7 \end{bmatrix}_{3x2}$ and $k = \frac{3}{2}$,

then $kA = \frac{3}{2}A$

$$= \frac{3}{2}\begin{bmatrix} -1 & 5 \\ 3 & -2 \\ 4 & 7 \end{bmatrix}_{3x2} = \begin{bmatrix} -\frac{3}{2} & \frac{15}{2} \\ \frac{9}{2} & -3 \\ 6 & \frac{21}{2} \end{bmatrix}_{3x2}$$

(3) Addition of Two matrices : A and B are two matrices of same order. Their addition denoted by A + B is a matrix obtained by adding the corresponding elements of A and B.

Note: A+B is possible only when Aand B are of same order.

A+B is of the same order as that of A and B.

Thus if $A = [a_{ij}]_{mxn}$ and $B = [b_{ij}]_{mxn}$ then $A+B = [a_{ij} + b_{ij}]_{mxn}$

Ex. $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -2 & 0 \end{bmatrix}_{2x3}$ and

$B = \begin{bmatrix} -4 & 3 & 1 \\ 5 & 7 & -8 \end{bmatrix}_{2x3}$ Find A+B.

Solution : By definition of addition,

$$A+B = \begin{bmatrix} 2+(-4) & 3+3 & 1+1 \\ -1+5 & -2+7 & 0+(-8) \end{bmatrix}_{2x3}$$

$$= \begin{bmatrix} -2 & 6 & 2 \\ 4 & 5 & -8 \end{bmatrix}_{2x3}$$

Note : If A and B are two matrices of the same order then subtraction of the two matrices is defined as, A–B = A+(–B), where –B is the negative of matrix B.

Ex. If $A = \begin{bmatrix} -1 & 4 \\ 3 & -2 \\ 0 & 5 \end{bmatrix}_{3x2}$ and $B = \begin{bmatrix} -1 & 5 \\ 2 & -6 \\ 4 & 9 \end{bmatrix}_{3x2}$, Find A–B.

Solution : By definition of subtraction,

$$A-B = A+(-B) = \begin{bmatrix} -1 & 4 \\ 3 & -2 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 1 & -5 \\ -2 & 6 \\ -4 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -1+1 & 4+(-5) \\ 3+(-2) & -2+6 \\ 0+(-4) & 5+(-9) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 4 \\ -4 & -4 \end{bmatrix}$$

Some Results on addition and scalar multiplication : If A, B, C are three matrices conformable for addition and α, β are scalars, then

(i) A+B = B+A, That is, the matrix addition is commutative.

(ii) (A+B)+C = A+(B+C), That is, the matrix addition is associative.

(iii) For matrix A, we have A+O = O+A = A, That is, a zero matrix is conformable for addition and it is the identity for matrix addition.

(iv) For a matrix A, we have A+(–A) = (–A) +A = O, where O is a zero matrix conformable with matrix A for addition. Where (–A) is additive inverse of A.

(v) $\alpha(A \pm B) = \alpha A \pm \alpha B$

(vi) $(\alpha \pm \beta)A = \alpha A \pm \beta A$

(vii) $\alpha(\beta \cdot A) = (\alpha \cdot \beta) \cdot A$

(viii) OA = O

SOLVED EXAMPLES

Ex. 1) If $A = \begin{bmatrix} 5 & -3 \\ 1 & 0 \\ -4 & -2 \end{bmatrix}$ and

$B = \begin{bmatrix} 2 & 7 \\ -3 & 1 \\ 2 & -2 \end{bmatrix}$, find 2A – 3B.

Solution : Let 2A – 3B

$$= 2\begin{bmatrix} 5 & -3 \\ 1 & 0 \\ -4 & -2 \end{bmatrix} - 3\begin{bmatrix} 2 & 7 \\ -3 & 1 \\ 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -6 \\ 2 & 0 \\ -8 & -4 \end{bmatrix} + \begin{bmatrix} -6 & -21 \\ 9 & -3 \\ -6 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 10-6 & -6-21 \\ 2+9 & 0-3 \\ -8-6 & -4+6 \end{bmatrix} = \begin{bmatrix} 4 & -27 \\ 11 & -3 \\ -14 & 2 \end{bmatrix}$$

Ex. 2) If A = diag(2, –5, 9), B = diag(–3, 7, –14) and C = diag(1, 0, 3), find B–A–C.

Solution : B–A–C = B–(A+C)

Now, A+C = diag(2, –5, 9) + diag(1, 0, 3)
= diag(3, –5, 12)

B–A–C= B–(A+C)
=diag(–3, 7, –14)–diag(3, –5, 12)

$$= \text{diag}(-6, 12, -26) = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & -26 \end{bmatrix}$$

Ex. 3) If $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 7 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & -1 \end{bmatrix}$ and

$C = \begin{bmatrix} 1 & -1 & 6 \\ 0 & 2 & -5 \end{bmatrix}$, find the matrix X such that

3A–2B+4X = 5C.

Solution : Since 3A–2B+4X = 5C

∴ 4X = 5C–3A+2B

$$\therefore 4X = 5\begin{bmatrix} 1 & -1 & 6 \\ 0 & 2 & -5 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 & -1 \\ 4 & 7 & 5 \end{bmatrix}$$

$$+2\begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -5 & 30 \\ 0 & 10 & -25 \end{bmatrix} + \begin{bmatrix} -6 & -9 & 3 \\ -12 & -21 & -15 \end{bmatrix}$$

$$+\begin{bmatrix} 2 & 6 & 4 \\ 8 & 12 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 5-6+2 & -5-9+6 & 30+3+4 \\ 0-12+8 & 10-21+12 & -25-15-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -8 & 37 \\ -4 & 1 & -42 \end{bmatrix}$$

$$\therefore X = \frac{1}{4}\begin{bmatrix} 1 & -8 & 37 \\ -4 & 1 & -42 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -2 & \frac{37}{4} \\ -1 & \frac{1}{4} & -\frac{21}{2} \end{bmatrix}$$

Ex. 4) If $\begin{bmatrix} 2x+1 & -1 \\ 3 & 4y \end{bmatrix} + \begin{bmatrix} -1 & 6 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 6 & 12 \end{bmatrix}$,

find x and y.

Solution : Given $\begin{bmatrix} 2x+1 & -1 \\ 3 & 4y \end{bmatrix} + \begin{bmatrix} -1 & 6 \\ 3 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 5 \\ 6 & 12 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2x & 5 \\ 6 & 4y \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 6 & 12 \end{bmatrix}$$

$\therefore$ Using definition of equality of matrices, we have

$2x = 4, \quad 4y = 12 \quad \therefore \; x = 2, \quad y = 3$

Ex. 5) If $X + Y = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ -3 & -2 \end{bmatrix}$ and

$X - 2Y$

$= \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 4 & -2 \end{bmatrix}$ then find X ,Y.

Solution : Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ -3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 4 & -2 \end{bmatrix}$

Let, $X + Y = A$ …. (1), $X - 2Y = B$ ……(2), Solving (1) and (2) for X and Y

By (1) – (2) , $3Y = A - B$, $\therefore Y = \frac{1}{3}(A - B)$

$$\therefore Y = \frac{1}{3}\left\{\begin{bmatrix} 2 & -1 \\ 1 & 3 \\ -3 & -2 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 4 & -2 \end{bmatrix}\right\} = \frac{1}{3}\begin{bmatrix} 4 & -2 \\ -2 & 4 \\ -7 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} \\ -\frac{7}{3} & 0 \end{bmatrix}$$

From (1) $X + Y = A$, $\therefore X = A - Y$,

$$\therefore X = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ -3 & -2 \end{bmatrix} - \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} \\ -\frac{7}{3} & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 2-\frac{4}{3} & -1+\frac{2}{3} \\ 1+\frac{2}{3} & 3-\frac{4}{3} \\ -3+\frac{7}{3} & -2+0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{5}{3} & \frac{5}{3} \\ -\frac{2}{3} & -2 \end{bmatrix}$$

EXERCISE 4.5

(1) If $A = \begin{bmatrix} 2 & -3 \\ 5 & -4 \\ -6 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 0 & 3 \end{bmatrix}$ and

$C = \begin{bmatrix} 4 & 3 \\ -1 & 4 \\ -2 & 1 \end{bmatrix}$ Show that (i) A +B = B+A

(ii) (A+B)+C = A+(B+C)

(2) If $A = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ 4 & -7 \end{bmatrix}$, then find the matrix A –2B+6I, where I is the unit matrix of order 2.

(3) If $A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 7 & -8 \\ 0 & -6 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 9 & -1 & 2 \\ -4 & 2 & 5 \\ 4 & 0 & -3 \end{bmatrix}$

then find the matrix C such that A+B+C is a zero matrix.

(4) If $A = \begin{bmatrix} 1 & -2 \\ 3 & -5 \\ -6 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 \\ 4 & 2 \\ 1 & 5 \end{bmatrix}$ and

$C == \begin{bmatrix} 2 & 4 \\ -1 & -4 \\ -3 & 6 \end{bmatrix}$, find the matrix X such that

3A –4B+5X = C.

(5) Solve the following equations for X and Y, if

$3X–Y= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $X–3Y= \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$

(6) Find matrices A and B, if

$2A–B = \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix}$ and

$A–2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$

(7) Simplify,

$$\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

(8) If $A = \begin{bmatrix} i & 2i \\ -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2i & i \\ 2 & -3 \end{bmatrix}$, where $\sqrt{-1} = i$, find A+B and A–B. Show that A+B is a singular. Is A–B a singular ? Justify your answer.

(9) Find x and y, if $\begin{bmatrix} 2x+y & -1 & 1 \\ 3 & 4y & 4 \end{bmatrix}$

$+ \begin{bmatrix} -1 & 6 & 4 \\ 3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 5 \\ 6 & 18 & 7 \end{bmatrix}$

(10) If $= \begin{bmatrix} 2a+b & 3a-b \\ c+2d & 2c-d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$, find a, b, c and d.

(11) There are two book shops owned by Suresh and Ganesh. Their sales (in Rupees) for books in three subject – Physics, Chemistry and Mathematics for two months, July and August 2017 are given by two matrices A and B.

July sales (in Rupees), Physics Chemistry Mathematics.

$A = \begin{bmatrix} 5600 & 6750 & 8500 \\ 6650 & 7055 & 8905 \end{bmatrix}$ First Row Suresh/ Second Row Ganesh

August sales (in Rupees), Physics Chemistry Mathematics

$B = \begin{bmatrix} 6650 & 7055 & 8905 \\ 7000 & 7500 & 10200 \end{bmatrix}$ First Row Suresh/ Second Row Ganesh then,

(i) Find the increase in sales in Rupees from July to August 2017.

(ii) If both book shops got 10 % profit in the month of August 2017, find the profit for each book seller in each subject in that month.

(4) Algebra of Matrices (continued)

Two Matrices A and B are said to be conformable for the product AB if the number of columns in A is equal to the number of rows in B. i.e. A is of order mxn and B is of order nxp.

In This case the product AB is amatrix defined as follows.

$$A_{m\times n} \times B_{n\times p} = C_{m\times p}, \text{ where } C_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

$$\text{If } A = [a_{ik}]_{m\times n} = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} \ldots & a_{1k} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2k} \ldots & a_{2n} \\ a_{31} & a_{32} \ldots & a_{3k} \ldots & a_{3n} \\ a_{i1} & a_{i2} \ldots & a_{ik} \ldots & a_{in} \\ a_{m1} & a_{m2} \ldots & a_{mk} \ldots & a_{mn} \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \to i^{th} \text{ row} \\ \\ \end{array}$$

$$B = [b_{kj}]_{n\times p} = \begin{bmatrix} b_{11} & b_{12} \ldots & b_{1j} \ldots & b_{1p} \\ b_{21} & b_{22} \ldots & b_{2j} \ldots & b_{2p} \\ b_{31} & b_{32} \ldots & b_{3j} \ldots & b_{3p} \\ b_{p1} & b_{p2} \ldots & b_{nj} \ldots & b_{np} \end{bmatrix}$$

$$\downarrow$$

$$j^{th} \text{ column}$$

then

$$C_{ij} = a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + \ldots\ldots + a_{in}\, b_{nj}$$

SOLVED EXAMPLES

Ex.1 : If $A = [a_{11}\ a_{12}\ a_{13}]$ and $B = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$

Find AB.

Solution : Since number of columns of A = number of rows of B = 3

Therefore product AB is defined and its order is 1. $(A)_{1\times 3}\,(B)_{3\times 1} = (AB)_{1\times 1}$

$$AB = [a_{11} \times b_{11} + a_{12} \times b_{21} + a_{13} \times b_{31}]$$

Ex.2 : Let $A = [1\ 3\ 2]_{1\times 3}$ and $B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}_{3\times 1}$, find AB.

Does BA exist? If yes, find it.

Solution : Product AB is defined and order of AB is 1.

$$\therefore\ AB = [1\ \ 3\ \ 2]\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = [1 \times 3 + 3 \times 2 + 2 \times 1]$$

$$= [11]_{1\times 1}$$

Again since number of column of B = number of rows of A=1

$\therefore$ The product BA also is defined and order of BA is 3.

$$BA = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}_{3\times 1} [1\ 3\ 2]_{1\times 3} = \begin{bmatrix} 3\times 1 & 3\times 3 & 3\times 2 \\ 2\times 1 & 2\times 3 & 2\times 2 \\ 1\times 1 & 1\times 3 & 1\times 2 \end{bmatrix}_{3\times 3}$$

$$= \begin{bmatrix} 3 & 9 & 6 \\ 2 & 6 & 4 \\ 1 & 3 & 2 \end{bmatrix}_{3\times 3}$$

Remark : Here AB and BA both are defined but AB $\neq$ BA.

Ex.3 : $A = \begin{bmatrix} -1 & -2 \\ -3 & 2 \\ 1 & 0 \end{bmatrix}_{3\times 2}$, $B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}_{2\times 2}$

Find AB and BA which ever exist.

Solution : Here A is order of 3×2 and B is of order 2×2. By conformability of product, AB is defined but BA is not defined.

$$\therefore\ AB = \begin{bmatrix} -1 & -2 \\ -3 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1+2 & -2+4 \\ -3-2 & -6-4 \\ 1+0 & 2+0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -5 & -10 \\ 1 & 2 \end{bmatrix}$$

Ex.4 : Let $A = \begin{bmatrix} 3 & 2 & -1 \\ -2 & 5 & 4 \end{bmatrix}_{2\times 3}$,

$$B = \begin{bmatrix} 3 & -3 \\ -4 & 2 \end{bmatrix}_{2\times 2}$$

Find AB and BA which ever exist.

Solution : Since number of columns of A ≠ number of rows of B. Product of AB is not defined. But number of columns of B = number of rows of A = 2, the product BA exists,

$$\therefore\ BA = \begin{bmatrix} 3 & -3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ -2 & 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9+6 & 6-15 & -3-12 \\ -12-4 & -8+10 & 4+8 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & -9 & -15 \\ -16 & 2 & 12 \end{bmatrix}$$

Ex.5 : Let $A = \begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix}$ Find AB and BA which ever exist.

Solution : Since A and B are two matrix of same order 2×2.

$\therefore$ Both the product AB and BA exist and are of same order 2×2

$$AB = \begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4-12 & 12+6 \\ -5+8 & 15-4 \end{bmatrix} = \begin{bmatrix} -16 & 18 \\ 3 & 11 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -4+15 & 3+6 \\ 16-10 & -12-4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 9 \\ 6 & -16 \end{bmatrix}$$

Here AB ≠ BA

Note :

From the above solved numerical Examples, for the given matrices A and B we note that,

i) If AB exists , BA may or may not exist.

ii) If BA exists , AB may or may not exist.

iii) If AB and BA both exist they may not be equal.

4.6 Properties of Matrix Multiplication :

1) For matrices A and B, matrix multiplication is not commutative that is AB ≠ BA.

2) For three matrices A,B,C. Matrix multiplication is associative. That is (AB)C = A(BC) if orders of matrices are suitable for multiplication.

e.g. Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \end{bmatrix}$,

$$C = \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}$$

Then $AB = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1+0 & -1-2 & 2+6 \\ 4+0 & -4-3 & 8+9 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 8 \\ 4 & -7 & 17 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 1 & -3 & 8 \\ 4 & -7 & 17 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2-9+0 & 1+3+16 \\ -8-21+0 & 4+7+34 \end{bmatrix} = \begin{bmatrix} -11 & 20 \\ -29 & 45 \end{bmatrix} \quad \text{.....(1)}$$

$$\therefore\ BC = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2-3+0 & 1+1+4 \\ 0-3+0 & 0+1+6 \end{bmatrix} = \begin{bmatrix} -5 & 6 \\ -3 & 7 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -5 & 6 \\ -3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -5-6 & 6+14 \\ -20-9 & 24+21 \end{bmatrix} = \begin{bmatrix} -11 & 20 \\ -29 & 45 \end{bmatrix} \quad .....(2)$$

From (1) and (2), (AB)C = A(BC)

3) For three matrices A,B,C, multiplication is distributive over addition.

i) A(B+C)= AB + AC
(left distributive law)

ii) (B+C)A= BA + CA
(right distributive law)

These laws can be verified by examples.

4) For a given square matrix A there exists a unit matrix I of the same order as that of A, such that AI=IA=A.

I is called Identity matrix for matrix multiplication.

e.g. Let $A = \begin{bmatrix} 3 & -2 & -1 \\ 2 & 0 & 4 \\ 1 & 3 & 2 \end{bmatrix}$,

and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\text{Then } AI = \begin{bmatrix} 3 & -2 & -1 \\ 2 & 0 & 4 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+0+0 & 0-2+0 & 0+0-1 \\ 2+0+0 & 0+0+0 & 0+0+4 \\ 1+0+0 & 0+3+0 & 0+0+2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & -1 \\ 2 & 0 & 4 \\ 1 & 3 & 2 \end{bmatrix} = IA = A$$

5) For any matrix A there exists a null matrix O such that a) AO = O and b) OA = O.

6) The product of two non zero matrices can be a zero matrix. That is AB = O but $A \neq O$, $B \neq O$

e.g. Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$,

Here $A \neq 0$, $B \neq 0$ but $AB = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$

$$\text{That is } AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

7) Positive integral powers of a square matrix A are obtained by repeated multiplication of A by itself. That is $A^2 = AA$, $A^3 = AAA$, ……,

$A^n = AA \ldots$ n times

(Activity)

If $A = \begin{bmatrix} 1 & -5 \\ 6 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 4 & 8 \end{bmatrix}$,

Find AB-2I, where I is unit matrix of order 2.

Solution : Given $A = \begin{bmatrix} 1 & -5 \\ 6 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 4 & 8 \end{bmatrix}$

$$\text{Consider } AB-2I = \begin{bmatrix} 1 & -5 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 8 \end{bmatrix} - 2\begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix}$$

$$\therefore AB-2I = \begin{bmatrix} \ldots\ldots & -3-40 \\ 12+28 & \ldots\ldots \end{bmatrix} - \begin{bmatrix} \ldots & 0 \\ 0 & \ldots \end{bmatrix}$$

$$= \begin{bmatrix} \ldots & -43 \\ 40 & \ldots \end{bmatrix} - \begin{bmatrix} \ldots & 0 \\ 0 & \ldots \end{bmatrix}$$

$$\therefore AB-2I = \begin{bmatrix} \ldots & -43 \\ 40 & \ldots \end{bmatrix}$$

SOLVED EXAMPLES

Ex. 1: If A= $\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \end{bmatrix}$, B = $\begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}$,

show that matrix AB is non singular.

Solution : let AB = $\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} -2-3+0 & 1+1+4 \\ 0-3+0 & 0+1+6 \end{bmatrix}$

$= \begin{bmatrix} -5 & 6 \\ -3 & 7 \end{bmatrix}$,

$\therefore |AB| = \begin{vmatrix} -5 & 6 \\ -3 & 7 \end{vmatrix}$

$= -35 + 18 = -17 \neq 0$

$\therefore$ matrix AB is nonsingular.

Ex. 2 : If A = $\begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ prove that A^2–5A is a scalar matrix.

Solution : Let A^2 = A.A

$= \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1+9+9 & 3+3+9 & 3+9+3 \\ 3+3+9 & 9+1+9 & 9+3+3 \\ 3+9+3 & 9+3+3 & 9+9+1 \end{bmatrix}$

$= \begin{bmatrix} 19 & 15 & 15 \\ 15 & 19 & 15 \\ 15 & 15 & 19 \end{bmatrix}$

$\therefore A^2 - 5A = \begin{bmatrix} 19 & 15 & 15 \\ 15 & 19 & 15 \\ 15 & 15 & 19 \end{bmatrix} - 5\begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 19 & 15 & 15 \\ 15 & 19 & 15 \\ 15 & 15 & 19 \end{bmatrix} - \begin{bmatrix} 5 & 15 & 15 \\ 15 & 5 & 15 \\ 15 & 15 & 5 \end{bmatrix}$

$\therefore A^2 - 5A = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = 14I$

$\therefore A^2$–5A is a scalar matrix.

Ex. 3 : If A= $\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, Find k, so that A^2 –kA+2I = O, where I is an identity matrix and O is null matrix of order 2.

Solution : Given A^2 –kA+2I = O

$\therefore$ Here, $A^2 = AA = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$

$= \begin{bmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$

$\therefore A^2$–kA+2I = O

$\therefore \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} - k\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = O$

$\therefore \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} - \begin{bmatrix} 3k & -2k \\ 4k & -2k \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore \begin{bmatrix} 1-3k+2 & -2+2k \\ 4-4k & -4+2k+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ Using definition of equality of matrices, we have

$$\left.\begin{array}{ll} 1-3k+2=0 & \therefore\ 3k=3 \\ -2+k=0 & \therefore\ 2k=2 \\ 4-4k=0 & \therefore\ 4k=4 \\ -4+2k+2=0 & \therefore\ 2k=2 \end{array}\right\} \quad k=1$$

Ex. 4 : Find x and y, if

$$[2\ \ 0\ \ 3]\left\{3\begin{bmatrix} 6 & 3 \\ -1 & 2 \\ 5 & 4 \end{bmatrix} + 2\begin{bmatrix} -4 & -1 \\ 1 & 0 \\ -3 & -4 \end{bmatrix}\right\} = [x\ \ y]$$

Solution :

Given $[2\ 0\ 3]\left\{3\begin{bmatrix}6 & 3\\ -1 & 2\\ 5 & 4\end{bmatrix}+2\begin{bmatrix}-4 & -1\\ 1 & 0\\ -3 & -4\end{bmatrix}\right\}=[x\quad y]$

$\therefore\ [2\ \ 0\ \ 3]\left\{\begin{bmatrix}18 & 9\\ -3 & 6\\ 15 & 12\end{bmatrix}+\begin{bmatrix}-8 & -2\\ 2 & 0\\ -6 & -8\end{bmatrix}\right\}=[x\quad y]$

$\therefore\ [2\ \ 0\ \ 3]\begin{bmatrix}10 & 7\\ -1 & 6\\ 9 & 4\end{bmatrix}=[x\quad y]$

$\therefore\ [20+27\ \ 14+12]=[x\quad y]$

$\therefore\ [47\ \ 26]=[x\quad y]$ $\therefore$ $x=47$, $y=26$ by definition of equality of matrices.

Ex. 5 : Find if $\begin{bmatrix}sin\theta\\ cos\theta\\ \theta\end{bmatrix}[sin\theta\quad cos\theta\quad \theta]=[17]$

Solution : Let $\begin{bmatrix}sin\theta\\ cos\theta\\ \theta\end{bmatrix}[sin\theta\quad cos\theta\quad \theta]=[17]$,

$\therefore\ [sin^2\theta+cos^2\theta+\theta^2]=[17]$

$\therefore$ By definition of equality of matries

$\therefore\ 1+\theta^2=17$ $\therefore\ \theta^2=17-1$ $\therefore\ \theta^2=16$,

$\therefore\ \theta=\pm\ 4$

Remark

Using the distributive laws discussed earlier we can derive the following results,

If A and B are square matrices of the same order, then

i) $(A+B)^2=A^2+AB+BA+B^2$

ii) $(A-B)^2=A^2-AB-BA+B^2$

iii) $(A+B)(A-B)=A^2+AB-BA-B^2$

Ex. 6 : If $A=\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}$, prove that $A^n=\begin{bmatrix}a^n & 0\\ 0 & b^n\end{bmatrix}$ for all n∈N.

Solution : Given $A=\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}$

We prove $A^n=\begin{bmatrix}a^n & 0\\ 0 & b^n\end{bmatrix}$ for all n∈N using mathematical induction

Let P(n) be $=\begin{bmatrix}a^n & 0\\ 0 & b^n\end{bmatrix}$ for n∈N.

To prove that P(n) is true for n=1

P(1) is $A^1=A=\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}$ $\therefore$ P(1) is true.

Assume that P(K) is true for some K∈N

That is P(K) is $A^k=\begin{bmatrix}a^K & 0\\ 0 & b^K\end{bmatrix}$

To prove that P(K) → P(K+1) is true consider L.H.S. of P(K+1)

That is A^{k+1}

$=A^k.A$

$=\begin{bmatrix}a^K & 0\\ 0 & b^K\end{bmatrix}\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}=\begin{bmatrix}a^{K+1}+0 & 0+0\\ 0+0 & b^{K+1}+0\end{bmatrix}$

$=\begin{bmatrix}a^{K+1} & 0\\ 0 & b^{K+1}\end{bmatrix}$ = R.H.S of P(K+1)

Hence P(K+1) is true.

$\therefore$ P(K) ⇒ P(K+1) for all K∈N

Hence by principle of mathematical induction, the statement P(n) is true for all n∈N.

That is P(n) is true → P(2) is true → P(3) is true and so on → P(n)is true, n∈N.

$\therefore\ A^n=\begin{bmatrix}a^n & 0\\ 0 & b^n\end{bmatrix}$ for all n∈N.

Ex. 7 : A school purchased 8 dozen Mathematics books, 7 dozen Physics books and 10 dozen Chemistry books, the prices are Rs.50,Rs.40 and Rs.60 per book respectively. Find the total amount that the book seller will receive from school authority using matrix multiplication.

Solution : Let A be the column matrix of books of different subjects and let B be the row matrix of prices of one book of each subject.

$$A = \begin{bmatrix} 8\times12 \\ 7\times12 \\ 10\times12 \end{bmatrix} = \begin{bmatrix} 96 \\ 84 \\ 120 \end{bmatrix} \qquad B = [50 \;\; 40 \;\; 60]$$

$\therefore$ The total amount received by the bookseller is obtained by matrix BA.

$$\therefore BA = [50 \;\; 40 \;\; 60] \begin{bmatrix} 96 \\ 84 \\ 120 \end{bmatrix}$$

$$= [50 \times 96 + 40 \times 84 + 60 \times 120]$$

$$= [4800 + 3360 + 7200] = [15360]$$

Thus the amount received by the bookseller from the school is Rs.15360.

Ex. 8 : Some schools send their students for extra training in Kabaddi, Cricket and Tennis to a sports standidium. There center charge fee is changed pen student for Coaching as well as 4 equipment and maintenances of the court. The information of students from each school is given below-

$$\begin{array}{l} \\ \text{Modern School} \\ \text{Progressive School} \\ \text{Sharada Sadan} \\ \text{Vidya Niketan} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Kabaddi} & \text{Cricket} & \text{Tennis} \end{array} \\ \begin{bmatrix} 20 & 35 & 15 \\ 18 & 36 & 12 \\ 24 & 12 & 8 \\ 25 & 20 & 6 \end{bmatrix} \end{array}$$

The charges per student for each game are given below-

$$\begin{array}{l} \\ \text{Kabaddi} \\ \text{Cricket} \\ \text{Tennis} \end{array} \begin{array}{c} \begin{array}{cc} \text{Coach} & \text{E \& M} \end{array} \\ \begin{bmatrix} 40 & 10 \\ 50 & 50 \\ 60 & 40 \end{bmatrix} \end{array}$$

E and M is for equipment and maintain

To find the expense of each school on Coaching and E and M can be fourd by multiplication of the above matrics.

$$\begin{array}{l} \\ \text{Modern} \\ \text{Progreesive} \\ \text{Sharda School} \\ \text{Vidya Niketan} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Kabaddi} & \text{Cricket} & \text{Tennis} \end{array} \\ \begin{bmatrix} 20 & 35 & 18 \\ 18 & 36 & 12 \\ 24 & 12 & 8 \\ 25 & 20 & 6 \end{bmatrix} \end{array} \text{X} \begin{array}{l} \\ \text{Kab} \\ \text{cri} \\ \text{Ten} \end{array} \begin{array}{c} \begin{array}{cc} \text{Coach} & \text{E \& M} \end{array} \\ \begin{bmatrix} 40 & 10 \\ 50 & 50 \\ 60 & 40 \end{bmatrix} \end{array}$$

$$\begin{array}{l} \text{Modern} \\ \text{Progressive} \\ \text{Sharada} \\ \text{Vidya} \end{array} \begin{bmatrix} 20 \times 40 + 36 \times 50 + 18 \times 60 & 20 \times 10 + 35 \times 50 + 18 \times 40 \\ 18 \times 40 + 36 \times 50 + 12 \times 60 & 18 \times 10 + 36 \times 50 + 12 \times 40 \\ 24 \times 40 + 12 \times 50 + 8 \times 60 & 24 \times 10 + 12 \times 50 + 8 \times 40 \\ 25 \times 40 + 20 \times 50 + 6 \times 60 & 25 \times 10 + 20 \times 50 + 6 \times 40 \end{bmatrix}$$

$$= \begin{bmatrix} 800 + 1750 + 108 & 200 + 1750 + 720 \\ 720 + 1800 + 720 & 180 + 1800 + 480 \\ 960 + 600 + 480 & 240 + 600 + 320 \\ 1000 + 1000 + 360 & 250 + 1000 + 240 \end{bmatrix} = \begin{bmatrix} 2650 & 2670 \\ 3240 & 2960 \\ 2040 & 1160 \\ 2360 & 1490 \end{bmatrix}$$

EXERCISE 4.6

1) Evaluate i) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & -4 & 3 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$

2) If $A = \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ 3 & -2 \end{bmatrix}$ show that $AB \neq BA$.

3) If $A = \begin{bmatrix} -1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & -3 & 1 \end{bmatrix}$,

$B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. State whether AB=BA?

Justify your answer.

4) Show that AB=BA where,

i) $A = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$

ii) $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $B = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

5) If $A = \begin{bmatrix} 4 & 8 \\ -2 & -4 \end{bmatrix}$, prove that $A^2 = 0$.

6) Verify A(BC) = (AB)C in each of the following cases.

i) $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ 0 & 3 \end{bmatrix}$ and

$C = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 0 & -2 \end{bmatrix}$

ii) $A = \begin{bmatrix} 2 & 4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 \\ 3 & 3 \\ -1 & 1 \end{bmatrix}$ and

$C = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

7) Verify that A(B+C)=AB+BC in each of the following matrices

i) $A = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$ and

$C = \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ -2 & 3 \\ 4 & 3 \end{bmatrix}$ and

$C = \begin{bmatrix} 1 & 2 \\ -2 & 0 \\ 4 & -3 \end{bmatrix}$

8) If $A = \begin{bmatrix} 1 & -2 \\ 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ 3 & 7 \end{bmatrix}$,

Find AB-2I,where I is unit matrix of order 2.

9) If $A = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & -2 \end{bmatrix}$ show that matrix AB is non singular.

10) If $A = \begin{bmatrix} 1 & 2 & 0 \\ 5 & 4 & 2 \\ 0 & 7 & -3 \end{bmatrix}$, find the product (A+I)(A–I).

11) $A = \begin{bmatrix} \propto & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ find α, if $A^2 = B$.

12) If A= $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, Show that $A^2 – 4A$ is a scalar matrix.

13) If A = $\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$, find k so that $A^2 – 8A – kI = O$, where I is a unit matrix and O is a null matrix of order 2.

14) If A = $\begin{bmatrix} 8 & 4 \\ 10 & 5 \end{bmatrix}$, B = $\begin{bmatrix} 5 & -4 \\ 10 & -8 \end{bmatrix}$ show that $(A + B)^2 = A^2 + AB + B^2$.

15) If A = $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, prove that $A^2 – 5A + 7I = 0$, where I is unit matrix of order 2.

16) If A = $\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$, show that $(A+B)(A-B)= A^2 – B^2$.

17) If A = $\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, B = $\begin{bmatrix} 2 & a \\ -1 & b \end{bmatrix}$ and if $(A + B)^2 = A^2+B^2$. find values of a and b.

18) Find matrix X such that AX=B, where A= $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} -3 \\ -1 \end{bmatrix}$.

19) Find k, if A= $\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and if $A^2 = kA - 2I$.

20) Find x, if $[1 \;\; x \;\; 1] \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = 0$.

21) Find x and y, if

$$\left\{4\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 4 \\ 2 & 1 & 1 \end{bmatrix}\right\} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

22) Find x, y, z if

$$\left\{3\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 3 & 1 \end{bmatrix}\right\} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x-3 \\ y-1 \\ 2z \end{bmatrix}.$$

23) If A = $\begin{bmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{bmatrix}$, show that

$$A^2 = \begin{bmatrix} cos2\alpha & sin2\alpha \\ -sin2\alpha & cos2\alpha \end{bmatrix}.$$

24) If A = $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, B = $\begin{bmatrix} 0 & 4 \\ 2 & -1 \end{bmatrix}$, show that $AB \neq BA$, but $|AB| = |A|.|B|$

25) Jay and Ram are two friends in a class. Jay wanted to buy 4 pens and 8 notebooks, Ram wanted to buy 5 pens and 12 notebooks. Both of them went to a shop. The price of a pen and a notebook which they have selected was Rs.6 and Rs.10. Using Matrix multiplication, find the amount required from each one of them.

4.7 Properties of transpose of a matrix :

Note :

(1) For any matrix A, $(A^T)^T = A$.

(2) If A is a matrix and k is constant, then $(kA)^T = kA^T$

(3) If A and B are two matrices of same order, then $(A + B)^T = A^T + B^T$

(4) If A and B are conformable for the product AB, then $(AB)^T = B^T A^T$

Example, Let A = $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}$, B = $\begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$,

$\therefore$ AB is defined and

$$AB = \begin{bmatrix} 2+2+1 & 3+4+2 \\ 6+1+3 & 9+2+6 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 10 & 17 \end{bmatrix},$$

$$\therefore (AB)^T = \begin{bmatrix} 5 & 10 \\ 9 & 17 \end{bmatrix} \quad \text{......... (1)}$$

$$\text{Now } A^T = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}, B^T = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix},$$

$$\therefore B^T A^T = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\therefore B^T A^T = \begin{bmatrix} 2+2+1 & 6+1+3 \\ 3+4+2 & 9+2+6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 \\ 9 & 17 \end{bmatrix} \quad \text{........ (2)}$$

$\therefore$ From (1) and (2) we have, $(AB)^T = B^T A^T$

In general $(A_1 A_2 A_3, \ldots\ldots A_n)^T = A_n^T \ldots A_3^T A_2^T A_1^T$

(5) If A is a symmetric matrix, then $A^T = A$.

For example, let $A = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 5 & -2 \\ 4 & -2 & 1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 5 & -2 \\ 4 & -2 & 1 \end{bmatrix} = A$$

(6) If A is a skew symmetric matrix, then $A^T = -A$.

For example, let $A = \begin{bmatrix} 0 & 5 & 4 \\ -5 & 0 & -2 \\ -4 & 2 & 0 \end{bmatrix}$

$$\therefore A^T = \begin{bmatrix} 0 & -5 & -4 \\ 5 & 0 & 2 \\ 4 & -2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 5 & 4 \\ -5 & 0 & -2 \\ -4 & 2 & 0 \end{bmatrix}$$

$$= -A, \quad \therefore A^T = -A$$

(7) If A is a square matrix, then (a) $A + A^T$ is symmetric. (b) $A - A^T$ is skew symmetric.

For example, (a) Let $A = \begin{bmatrix} 3 & 5 & 7 \\ 2 & 4 & -6 \\ 3 & 8 & -5 \end{bmatrix}$,

$$\therefore A^T = \begin{bmatrix} 3 & 2 & 3 \\ 5 & 4 & 8 \\ 7 & -6 & -5 \end{bmatrix}$$

$$\text{Now } A + A^T = \begin{bmatrix} 3 & 5 & 7 \\ 2 & 4 & -6 \\ 3 & 8 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 3 \\ 5 & 4 & 8 \\ 7 & -6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 7 & 10 \\ 7 & 8 & 2 \\ 10 & 2 & -10 \end{bmatrix}$$

$\therefore A + A^T$ is a symmetric matrix, by definition.

$$\text{(b) Let } A - A^T = \begin{bmatrix} 3 & 5 & 7 \\ 2 & 4 & -6 \\ 3 & 8 & -5 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 3 \\ 5 & 4 & 8 \\ 7 & -6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 & 4 \\ -3 & 0 & -14 \\ -4 & 14 & 0 \end{bmatrix}$$

$A - A^T$ is a skew symmetric matrix, by definition.

Note:

A square matrix A can be expressed as the sum of a symmetric and a skew symmetric matrix as $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

e.g. Let $A = \begin{bmatrix} 4 & -5 & 3 \\ -6 & 2 & 1 \\ 7 & 8 & -9 \end{bmatrix}$,

$$\therefore A^T = \begin{bmatrix} 4 & -6 & 7 \\ -5 & 2 & 8 \\ 3 & 1 & -9 \end{bmatrix}$$

$$A + A^T = \begin{bmatrix} 4 & -5 & 3 \\ -6 & 2 & 1 \\ 7 & 8 & -9 \end{bmatrix} + \begin{bmatrix} 4 & -6 & 7 \\ -5 & 2 & 8 \\ 3 & 1 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -11 & 10 \\ -11 & 4 & 9 \\ 10 & 9 & -18 \end{bmatrix}$$

$$\text{Let } P = \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 8 & -11 & 10 \\ -11 & 4 & 9 \\ 10 & 9 & -18 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -\frac{11}{2} & 5 \\ -\frac{11}{2} & 2 & \frac{9}{2} \\ 5 & \frac{9}{2} & -9 \end{bmatrix}$$

The matrix P is a symmetric matrix.

$$\text{Also } A - A^T = \begin{bmatrix} 4 & -5 & 3 \\ -6 & 2 & 1 \\ 7 & 8 & -9 \end{bmatrix} - \begin{bmatrix} 4 & -6 & 7 \\ -5 & 2 & 8 \\ 3 & 1 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -4 \\ -1 & 0 & -7 \\ 4 & 7 & 0 \end{bmatrix}$$

$$\text{Let } Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 0 & 1 & -4 \\ -1 & 0 & -7 \\ 4 & 7 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{2} & -2 \\ -\frac{1}{2} & 0 & -\frac{7}{2} \\ 2 & \frac{7}{2} & 0 \end{bmatrix}$$

The matrix Q is a skew symmetric matrix.

Since P+Q = symmetric matrix + skew symmetric matrix.

Thus A = P + Q.

EXERCISE 4.7

(1) Find A^T, if (i) $A = \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 2 & -6 & 1 \\ -4 & 0 & 5 \end{bmatrix}$

(2) If $[a_{ij}]_{3x3}$ where $a_{ij} = 2(i–j)$. Find A and A^T. State whether A and A^T are symmetric or skew symmetric matrices ?

(3) If $A = \begin{bmatrix} 5 & -3 \\ 4 & -3 \\ -2 & 1 \end{bmatrix}$, Prove that $(2A)^T = 2A^T$.

(4) If $A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ -5 & 4 & 9 \end{bmatrix}$, Prove that $(3A)^T = 3A^T$.

(5) If $A = \begin{bmatrix} 0 & 1+2i & i-2 \\ -1-2i & 0 & -7 \\ 2-i & 7 & 0 \end{bmatrix}$

where $i = \sqrt{-1}$, Prove that $A^T = –A$.

(6) If $A = \begin{bmatrix} 2 & -3 \\ 5 & -4 \\ -6 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 4 & -1 \\ -3 & 3 \end{bmatrix}$ and

$C = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ -2 & 3 \end{bmatrix}$ then show that

(i) $(A + B) = A^T + B^T$ (ii) $(A - C)^T = A^T - C^T$

(7) If $A = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 4 & -1 \end{bmatrix}$, then find C^T, such that $3A - 2B + C = I$, where I is the unit matrix of order 2.

(8) If $A = \begin{bmatrix} 7 & 3 & 0 \\ 0 & 4 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 1 & -4 \end{bmatrix}$ then find (i) $A^T + 4B^T$ (ii) $5A^T - 5B^T$.

(9) If $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & -4 \\ 3 & 5 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 2 & 3 \\ -1 & -1 & 0 \end{bmatrix}$, verify that $(A + 2B + 2C)^T = A^T + 2B^T + 3C^T$.

(10) If $A = \begin{bmatrix} -1 & 2 & 1 \\ -3 & 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \\ -1 & 3 \end{bmatrix}$, prove that $(A + B^T)^T = A^T + B$.

(11) Prove that $A + A^T$ is a symmetric and $A - A^T$ is a skew symmetric matrix, where

(i) $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -2 & -3 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 5 & 2 & -4 \\ 3 & -7 & 2 \\ 4 & -5 & -3 \end{bmatrix}$

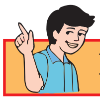
(12) Express the following matrices as the sum of a symmetric and a skew symmetric matrix.

(i) $\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

(13) If $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 & -4 \\ 2 & -1 & 1 \end{bmatrix}$, verify that (i) $(AB)^T = B^T A^T$

(ii) $(BA)^T = A^T B^T$

(14) If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that $A^T A = I$, where I is the unit matrix of order 2.



Let's Remember

- The value of a determinant of order 3 × 3

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

- The minors and cofactors of elements of a determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Minor M_{ij} of the element a_{ij} is determinant obtained by deleting the i^{th} row and j^{th} column of determinant D. The cofactor C_{ij} of element a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$

- **Properties of determinant**

Property (i) - The value of determinant remains unchanged if its rows are turned into columns and columns are turned into rows.

Property (ii) - If any two rows (or columns) of the determinant are interchanged then the value of determinant changes its sign.

Property (iii) - If any two rows (or columns) of a determinant are identical then the value of determinant is zero

Property (iv) - If any element of a row (or column) of determinant is multiplied by a constant k then the value of the new determinant is k times the value of old determinant

Property (v) - If each element of a row (or column) is expressed as the sum of two numbers then the determinant can be expressed as sum of two determinants

Property (vi) - If a constant multiple of all elements of a row (or column) is added to the corresponding elements of any other row (or column) then the value of new determinant so obtained is the same as that of the original determinant.

Property (vii) - (Triangle property) - If each element of a determinant above or below the main diagonal is zero then the value of the determinant is equal to the product of its diagonal elements.

- A system of linear equations, using Cramer's Rule has solution -

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}\text{ ; provided } D \neq 0$$

- Consistency of three equations.

$a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$

$a_3x + b_3y + c_3 = 0$ are consistent

$$\text{Then } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

- Area of triangle whose vertices are (x_1, y_1), (x_2,y_2) (x_3,y_3) is

$$A(\Delta) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- Test for collinear of points (x_1,y_1), (x_2,y_2), (x_3,y_3) if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

- **Multiplication of a matrix by a scalar:**

If A $[a_{ij}]$ is a matrix and k is a scalar, then kA = $[ka_{ij}]$.

- **Addition of matrices:**

Matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ are said to conformable for addition if orders of A and B are same.

A+B = $[a_{ij} + b_{ij}]$. The order of A+B is the same as that of A and B.

- **Multiplication of two matrices:**

A and B are said to be conformable for the multiplication if number of columns of A is equal to the number of rows of B.

That is If A = $[a_{ik}]_{m\times p}$ and B = $[b_{kj}]_{p\times n}$, then AB is defined and AB = $[c_{ij}]_{m\times n}$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}.b_{kj} \qquad \begin{array}{l} i = 1,2,\ldots.,m \\ j = 1,2,\ldots..,n. \end{array}$$

- If A = $[a_{ij}]_{m\times n}$ is any matrix, then the transpose of A is denoted by $A^T = B = [b_{ij}]_{n\times m}$ and $b_{ij} = a_{ji}$

- If A is a square matrix, then

i) $A + A^T$ is a symmetric matrix.

ii) $A - A^T$ is a skew-symmetric matrix.

- Every square matrix A can be expressed as the sum of a symmetric and skew-symmetric matrix as

$$A = \frac{1}{2}[A + A^T] + \frac{1}{2}[A - A^T].$$

MISCELLANEOUS EXERCISE - 4 (B)

(I) Select the correct option from the given alternatives.

1) Given $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ if $A - \lambda I$ is a singular matrix then ……….

A) $\lambda = 0$ B) $\lambda^2 - 3\lambda - 4 = 0$

C) $\lambda^2 + 3\lambda - 4 = 0$ D) $\lambda^2 - 3\lambda - 6 = 0$

2) Consider the matrices $A = \begin{bmatrix} 4 & 6 & -1 \\ 3 & 0 & 2 \\ 1 & -2 & 5 \end{bmatrix}$,

$B = \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ out of the given matrix product ………..

i) $(AB)^TC$ ii) $C^TC(AB)^T$

iii) C^TAB iv) A^TABB^TC

A) Exactly one is defined

B) Exactly two are defined

C) Exactly three are defined

D) all four are defined

3) If A and B are square matrices of equal order, then which one is correct among the following?

A) $A + B = B + A$ B) $A + B = A - B$

C) $A - B = B - A$ D) $AB = BA$

4) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is the identity matrix of order 3, then the ordered pair (a, b) is equal to …….

A) (2, –1) B) (–2, 1)

C) (2, 1) D) (–2, –1)

5) If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$, then $\alpha =$ …….

A) ±3 B) ±2 C) ±5 D) 0

6) If $\begin{bmatrix} 5 & 7 \\ x & 1 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ -3 & 5 \\ 2 & y \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 4 & -4 \\ 0 & 4 \end{bmatrix}$ then ……

A) $x = 1, y = -2$ B) $x = -1, y = 2$

C) $x = 1, y = 2$ D) $x = -1, y = -2$

7) If $A + B = \begin{bmatrix} 7 & 4 \\ 8 & 9 \end{bmatrix}$ and $A - B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ then the value of A is …….

A) $\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$ B) $\begin{bmatrix} 4 & 3 \\ 4 & 6 \end{bmatrix}$

C) $\begin{bmatrix} 6 & 2 \\ 8 & 6 \end{bmatrix}$ D) $\begin{bmatrix} 7 & 6 \\ 8 & 12 \end{bmatrix}$

8) If $\begin{bmatrix} x & 3x - y \\ zx + z & 3y - w \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$ then ……….

a) $x = 3, y = 7, z = 1, w = 14$

a) $x = 3, y = -5, z = -1, w = -4$

a) $x = 3, y = 6, z = 2, w = 7$

a) $x = -3, y = -7, z = -1, w = -14$

9) For suitable matrices A, B, the false statement is …….

A) $(AB)^T = A^TB^T$

B) $(A^T)^T = A$

C) $(A - B)^T = A^T - B^T$

D) $(A + B)^T = A^T + B^T$

10) If $A = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}$ and $f(x) = 2x^2 - 3x$, then $f(A) =$ ………

A) $\begin{bmatrix} 14 & 1 \\ 0 & -9 \end{bmatrix}$ B) $\begin{bmatrix} -14 & 1 \\ 0 & 9 \end{bmatrix}$

C) $\begin{bmatrix} 14 & -1 \\ 0 & 9 \end{bmatrix}$ D) $\begin{bmatrix} -14 & -1 \\ 0 & -9 \end{bmatrix}$

(II) Answer the following question.

1) If A = diag [2 –3 –5], B = diag [4 –6 –3] and C= diag [–3 4 1] then find i) B + C – A ii) 2A + B – 5C.

2) If f (α) = A = $\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, Find

i) f($-\alpha$) ii) f($-\alpha$) + f(α).

3) Find matrices A and B, where

i) 2A – B= $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and A +3 B= $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

ii) 3A – B= $\begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 5 \end{bmatrix}$ and

A + 5B= $\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$

4) If A = $\begin{bmatrix} 2 & -3 \\ 3 & -2 \\ -1 & 4 \end{bmatrix}$, B = $\begin{bmatrix} -3 & 4 & 1 \\ 2 & -1 & -3 \end{bmatrix}$

Verify i) $(A + B^T)^T = A^T + 2B$.

ii) $(3A - 5B^T)^T = 3A^T - 5B$.

5) If A = $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ and A + A^T = I, where I is unit matrix 2 × 2, then find the value of α.

6) If A= $\begin{bmatrix} 1 & 2 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 3 & 2 \\ 4 & -1 & -3 \end{bmatrix}$, show that AB is singular.

7) If A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$, B= $\begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$, show that AB and BA are both singular matrices.

8) If A= $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$, B= $\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$,

show that BA=6I.

9) If A = $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, B= $\begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}$, verify that |AB|= |A||B|.

10) If $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$,

show that $A_\alpha . A_\beta = A_{\alpha+\beta}$.

11) If A = $\begin{bmatrix} 1 & \omega \\ \omega^2 & 1 \end{bmatrix}$, B = $\begin{bmatrix} \omega^2 & 1 \\ 1 & \omega \end{bmatrix}$, where ω is a complex cube root of unity, then show that AB+BA+A–2B is a null matrix.

12) If A = $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ show that A^2 =A.

13) If A = $\begin{bmatrix} 4 & -1 & -4 \\ 3 & 0 & -4 \\ 3 & -1 & -3 \end{bmatrix}$, show that A^2 = I.

14) If A = $\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, show that $A^2 - 5A - 14I = 0$.

15) If A = $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 4A + 3I = 0$.

16) If A = $\begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix}$, B = $\begin{bmatrix} 1 & x \\ y & 0 \end{bmatrix}$, and

$(A + B)(A - B) = A^2 - B^2$, find x and y.

17) If $A=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ show that $(A+B)(A-B)\neq A^2-B^2$.

18) If $A=\begin{bmatrix}2 & -1\\3 & -2\end{bmatrix}$, find A^3.

19) Find x, y if,

i) $[0\ \ -1\ \ 4]\left\{2\begin{bmatrix}4 & 5\\3 & 6\\2 & -1\end{bmatrix}+3\begin{bmatrix}4 & 3\\1 & 4\\0 & -1\end{bmatrix}\right\}=[x\quad y]$.

ii)

$$\left\{-1\begin{bmatrix}1 & 2 & 1\\2 & 0 & 3\end{bmatrix}+3\begin{bmatrix}2 & -3 & 7\\1 & -1 & 3\end{bmatrix}\right\}\begin{bmatrix}5\\0\\-1\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$$

20) Find x, y, z if

i) $\left\{5\begin{bmatrix}0 & 1\\1 & 0\\1 & 1\end{bmatrix}-3\begin{bmatrix}2 & 1\\3 & -2\\1 & 3\end{bmatrix}\right\}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}x-1\\y+1\\2z\end{bmatrix}$

ii) $\left\{\begin{bmatrix}1 & 3 & 2\\2 & 0 & 1\\3 & 1 & 2\end{bmatrix}+2\begin{bmatrix}3 & 0 & 2\\1 & 4 & 5\\2 & 1 & 0\end{bmatrix}\right\}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}x\\y\\z\end{bmatrix}$

21) If $A=\begin{bmatrix}2 & 1 & -3\\0 & 2 & 6\end{bmatrix}$, $B=\begin{bmatrix}1 & 0 & -2\\3 & -1 & 4\end{bmatrix}$,

find AB^T and A^TB.

22) If $A=\begin{bmatrix}2 & -4\\3 & -2\\0 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & -1 & 2\\-2 & 1 & 0\end{bmatrix}$,

show that $(AB)^T=B^TA^T$.

23) If $A=\begin{bmatrix}3 & -4\\1 & -1\end{bmatrix}$, prove that

$A^n=\begin{bmatrix}1+2n & -4n\\n & 1-2n\end{bmatrix}$, for all $n\in N$.

24) If $A=\begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}$, prove that

$A^n=\begin{bmatrix}\cos n\theta & \sin n\theta\\-\sin n\theta & \cos n\theta\end{bmatrix}$, for all $n\in N$.

25) Two farmers Shantaram and Kantaram cultivate three crops rice,wheat and groundnut. The sale (In Rupees) of these crops by both the farmers for the month of April and may 2008 is given below,

April sale (In Rs.)

	Rice	Wheat	Groundnut
Shantaram	15000	13000	12000
Kantaram	18000	15000	8000

May sale (In Rs.)

	Rice	Wheat	Groundnut
Shantaram	18000	15000	12000
Kantaram	21000	16500	16000

Find

i) The total sale in rupees for two months of each farmer for each crop.

ii) the increase in sale from April to May for every crop of each farmer.