4. SEQUENCES AND SERIES

Let's study.

- Geometric progression (G.P.)
- n^{th} term of a G.P.
- Sum of n terms of a G.P.
- Sum of infinite terms of a G.P.
- Sigma notation.

4.1 SEQUENCE :

A set of numbers, where the numbers are arranged in a definite order, is called a sequence.

Natural numbers is an example of a sequence.

In general,a sequence is written as $\{t_n\}$.

Finite sequence – A sequence containing finite number of terms is called a finite sequence.

Infinite sequence – A sequence is said to be infinite if it is not a finite sequence.

In this case for every positive integer n , there is a unique t_n in the sequence.

Sequences that follow specific rule are called progressions.

In the previous class, we have studied Arithmetic Progression (A.P.).

In a sequence if the difference between any term and its preceding term $(t_{n+1}-t_n)$ is constant, for all $n \in N$ then the sequence is called an Arithmetic Progression (A.P.)

Consider the following sequences.

1) 2,5,8,11,14,…………..

2) 4,10,16,22,28,…………..

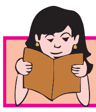
3) 4,16,64,256,…………….

4) $\frac{1}{5}, \frac{1}{25}, \frac{1}{125}$,…………..

The sequences 1) and 2), are in A.P. because $t_{n+1}-t_n$ is constant.

But in sequences 3) and 4), the difference is not constant. In these sequences, the ratio of any term to its preceding term that is $\frac{t_{n+1}}{t_n}$ is constant $[n \in N]$.

Such a sequence is called a 'GEOMETRIC PROGRESSION' (G. P.).



Let's learn.

4.2 GEOMETRIC PROGRESSION (G.P.)

Definition : A Sequence $\{t_n\}$ is said to be a Geometric Progression if $\frac{t_{n+1}}{t_n}$ = constant.

$\frac{t_{n+1}}{t_n}$ is called the common ratio of the G.P. and it is denoted by r $(r \neq 0)$, for all $n \in N$.

It is a convention to denote the first term of the geometric progression by a $(a \neq 0)$.

The terms of a geometric progression with first term 'a' and common ratio 'r' are as follows.

a, ar, ar^2, ar^3, ar^4,……………………

Let's see some examples of G.P.

(i) 2 , 8 , 32 , 128 , 512 , ………………… is a G.P. with a = 2 and r = 4.

(ii) 25, 5 , 1 , $\frac{1}{5}$, ………….................is a G.P. with a = 25 and r = $\frac{1}{5}$.

4.3 General term or the n^{TH} term of a G.P.

If a and r are the first term and common ratio of G.P. respectively, then its general term is given by $t_n = ar^{n-1}$.

Let's find n^{th} term of the following G.P.

i) 2,8,32,128,512,…………………

Here a=2 , r = 4

$t_n = ar^{n-1} = 2\,(4)^{n-1}$

ii) 25, 5 , 1 ,$\frac{1}{5}$,………….. ....

Here a = 25 , r = $\frac{1}{5}$

$t_n = ar^{n-1} = 25\,(\frac{1}{5})^{n-1}$

4.3.1 Properties of Geometric Progression.

i) Reciprocals of terms of a G.P. are also in G.P.

ii) If each term of a G.P. is multiplied or divided by a non zero constant ,then the resulting sequence is also a G.P.

iii) If each term of a G.P. is raised to the same power , the resulting sequence is also a G.P.

SOLVED EXAMPLES

Ex.1) For the following G.P.s find the n^{th} term

3,-6,12,-24,……………….

Solution:

Here a=3 , r = –2

$\therefore\ t_n = ar^{n-1} = 3\,(-2)^{n-1}$

Ex 2) Verify whether 1, $\frac{-3}{2}$, $\frac{9}{4}$, $\frac{-27}{8}$, ………… is a G.P. If it is a G.P., find its ninth term.

Solution : Here $t_1 = 1$, $t_2 = \frac{-3}{2}$, $t_3 = \frac{9}{4}$,

$$\text{Consider } = \frac{t_2}{t_1} = \frac{\frac{-3}{2}}{1} = \frac{-3}{2}$$

$$\frac{t_3}{t_2} = \frac{\frac{9}{4}}{\frac{-3}{2}} = \frac{-3}{2}$$

$$\frac{t_4}{t_3} = \frac{\frac{-27}{8}}{\frac{9}{4}} = \frac{-3}{2}$$

Here the ratio of any term to its previous term is constant hence the given sequence is a G.P.

$$\text{Now } t_9 = ar^{9-1} = a\,r^{8} = 1\left(\frac{-3}{2}\right)^{8} = \frac{6561}{256}.$$

Ex 3) For a G.P. if $a = 3$ and $t_7 = 192$, find r and t_{11} .

Solution : Given $a = 3$, $t_7 = ar^6 = 192$

$\therefore\ 3\,(r)^6 = 192$, $r^6 = \frac{192}{3} = 64$

$\therefore\ r^6 = 2^6$,

$\therefore\ r = \pm 2.$

$t_{11} = a\,r^{10} = 3\,(\pm 2)^{10} = 3(1024) = 3072.$

Ex 4) For a G.P. $t_3 = 486$, $t_6 = 18$, find t_{10}

Solution : We know that $t_n = ar^{n-1}$

$t_3 = ar^2 = 486$ ----------------- (1)

$t_6 = ar^5 = 18$ ----------------- (2)

dividing equation (2) by (1) we get,

$$\frac{t_6}{t_3} = \frac{ar^5}{ar^2} = \frac{18}{486} = \frac{2}{54} = \frac{1}{27}$$

$$r^3 = \left(\frac{1}{3}\right)^3 , \quad r = \frac{1}{3}.$$

Now from (1) $ar^2 = 486$

$$a\left(\frac{1}{3}\right)^2 = 486.$$

$$a\left(\frac{1}{9}\right) = 486,$$

$$a = 486 \times 9$$

$$a = 4374.$$

$$\text{Now } t_{10} = ar^9 = 4374\left(\frac{1}{3}\right)^9$$

$$= \frac{243 \times 2 \times 9}{3^9} = \frac{3^5 \times 2 \times 3^2}{3^5 \times 3^2 \times 3^2}$$

$$\therefore\ t_{10} = \frac{2}{9}$$

Ex 5) If for a sequence $\{t_n\}$, $t_n = \frac{5^{n-2}}{4^{n-3}}$, show that the sequence is a G.P. Find its first term and the common ratio.

Solution: $t_n = \frac{5^{n-2}}{4^{n-3}}$

$$t_{n+1} = \frac{5^{n-1}}{4^{n-2}}$$

$$\text{Consider } \frac{t_{n+1}}{t_n} = \frac{\frac{5^{n-1}}{4^{n-2}}}{\frac{5^{n-2}}{4^{n-3}}}$$

$$= \frac{5^{n-1}}{4^{n-2}} \times \frac{4^{n-3}}{5^{n-2}} = \frac{5}{4} = \text{constant},$$

$\forall\ n \in N.$

The given sequence is a G.P. with $r = \frac{5}{4}$ and $t_1 = a = \frac{5^{1-2}}{4^{1-3}} = \frac{5^{-1}}{4^{-2}} = \frac{16}{5}$.

Let's Note.

i) To find 3 numbers in G.P., it is convenient to take the numbers as $\frac{a}{r}, a, ar$

ii) 4 numbers in a G.P. as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$, (here the ratio is r^2)

iii) 5 numbers in a G.P. as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

Ex 6) Find three numbers in G.P. such that their sum is 42 and their product is 1728.

Solution: Let the three numbers be $\frac{a}{r}, a, ar$.

According to first condition their sum is 42

$$\therefore\ \frac{a}{r} + a + ar = 42$$

$$a\left(\frac{1}{r} + 1 + r\right) = 42$$

$$\therefore\ \frac{1}{r} + 1 + r = \frac{42}{a}$$

$$\frac{1}{r} + r = \frac{42}{a} - 1 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

From the second condition their product is 1728

$$\frac{a}{r} \cdot a \cdot ar = 1728$$

$$\therefore\ a^3 = 1728 = (12)^3$$

$$\therefore\ a = 12.$$

substitute a = 12 in equation (1), we get

$$\frac{1}{r} + r = \frac{42}{12} - 1$$

$$\frac{1}{r} + r = \frac{42-12}{12}$$

$$\frac{1}{r} + r = \frac{30}{12}$$

$$\frac{1+r^2}{r} = \frac{5}{2}$$

$$\therefore\ 2 + 2r^2 = 5r$$

$$\therefore\ 2r^2 - 5r + 2 = 0$$

$$\therefore\ (2r-1)(r-2) = 0$$

$$\therefore\ 2r = 1 \text{ or } r = 2$$

$$\therefore\ r = \frac{1}{2} \text{ or } r = 2$$

Now if a=12, and $r = \frac{1}{2}$ then the required numbers are 24 , 12 , 6.

If $a = 12$, and $r = 2$ then the required numbers are 6,12,24.

$\therefore$ 24,12,6 or 6,12,24 are the three required numbers in G.P.

Ex 7) In a G.P. ,if the third term is $\frac{1}{5}$ and sixth term is $\frac{1}{625}$, find its n^{th} term .

Solution : Here $t_3 = \frac{1}{5}$, $t_6 = \frac{1}{625}$

$$t_3 = ar^2 = \frac{1}{5} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1) \ ,$$

$$t_6 = ar^5 = \frac{1}{625} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Divide equation (2) by equation (1)

we get,

$$r^3 = \frac{1}{125} = \frac{1}{5^3}$$

$\therefore r = \frac{1}{5}$. Substitute $r = \frac{1}{5}$ in equation (1)

we get

$$a\left(\frac{1}{5}\right)^2 = \frac{1}{5}$$

$\therefore$ a = 5.

$$t_n = ar^{n-1} = 5\left(\frac{1}{5}\right)^{n-1} = 5 \text{ x } 5^{1-n} = 5^{2-n}.$$

Ex 8) Find four numbers in G. P. such that their product is 64 and sum of the second and third number is 6.

Solution : Let the four numbers be $\frac{a}{r^3}$, $\frac{a}{r}$, ar, ar^3 (common ratio is r^2)

According to the first condition

$$\frac{a}{r^3} \times \frac{a}{r} \times ar \times ar^3 = 64$$

$\therefore a^4 = 64$

$\therefore a = 2\sqrt{2}$.

Now using second condition $\frac{a}{r} + ar = 6$

$\frac{2\sqrt{2}}{r} + 2\sqrt{2}r = 6$. dividing by 2 we get ,

$\frac{\sqrt{2}}{r} + \sqrt{2}\ r = 3$ now multiplying by r we get

$$\sqrt{2} + \sqrt{2}\ r^2 - 3r = 0$$

$$\sqrt{2}\ r^2 - 3r + \sqrt{2} = 0 ,$$

$$\sqrt{2}\ r^2 - 2r - r + \sqrt{2} = 0,$$

$$\sqrt{2}\ r\,(r - \sqrt{2}) - 1\,(r - \sqrt{2}) = 0.$$

$$r = \sqrt{2} \text{ or } r = \frac{1}{\sqrt{2}} \ .$$

If a = $2\sqrt{2}$, and r = $\sqrt{2}$ then 1 ,2 , 4, 8 are the four required numbers

If a = $2\sqrt{2}$, and r = $\frac{1}{\sqrt{2}}$ then 8 ,4, 2, 1 are the four required numbers in G.P.

Ex 9) If p,q,r,s are in G.P. then show that

$$(q-r)^2 + (r-p)^2 + (s-q)^2 = (p-s)^2$$

Solution : As p,q,r,s are in G.P. $\frac{q}{p} = \frac{r}{q} = \frac{s}{r} = k$ (say)

$\therefore q^2 = pr$, $r^2 = qs$, $qr = ps$

consider L.H.S.

$= (q-r)^2 + (r-p)^2 + (s-q)^2$

$= q^2 - 2qr + r^2 + r^2 - 2rp + p^2 + s^2 - 2sq + q^2$

$= pr - 2qr + qs + qs - 2rp + p^2 + s^2 - 2sq + pr$

$= -2qr + p^2 + s^2 = -2ps + p^2 + s^2$ ($qr = ps$)

$= (p-s)^2$ = R.H.S.

Ex 10) Shraddha deposited Rs. 8000 in a bank which pays annual interest rate of 8%.She kept it with the bank for 10 years with compound interest. Find the total amount she will receive after 10 years . [given $(1.08)^{10} = 2.1589$)]

Solution:

The Amount deposited in a bank is Rs 8000 with 8% compound interest.

Each year, the ratio of the amount to the principal of that year is constant = $\frac{108}{100}$

Hence we get a G.P. of successive amounts.

For P = 8000,

the amount after 1 year is $8000 \times \frac{108}{100}$

the amount after 2 years is $8000 \times \frac{108}{100} \times \frac{108}{100}$

the amount after 3 years is $8000 \times \frac{108}{100} \times \frac{108}{100} \times \frac{108}{100}$.

Therefore after 10 years the amount is

$8000\left(\frac{108}{100}\right)^{10} = 8000\,(1.08)^{10}$.

$= 8000 \times 2.1589 = 17271$

Thus Shraddha will get Rs 17271 after 10 years.

Ex 11) The number of bacteria in a culture doubles every hour. If there were 50 bacteria originally in the culture, how many bacteria will be there after 5 hours ?

Solution : Given that the number of bacteria doubles every hour .

The ratio of bacteria after 1 hour to that at the begining is 2

after 1 hour = 50 x 2

after 2 hour = 50 x 2^2

after 3 hour = 50 x 2^3

Hence it is a G.P. with a=50 , r = 2 .

To find the number of bacteria present after 5 hours, that is to find t_6 .

$t_6 = ar^5 = 50\,(2)^5 = 50\,(32) = 1600$

ACTIVITIES

Activity 4.1:

Verify whether 1, $\frac{-4}{3}$, $\frac{16}{9}$, $\frac{-64}{27}$, ... is a G.P.

If it is a G.P. Find its ninth term.

Solution : Herer $t_1 = 1$, $t_2 = \boxed{}$, $t_3 = \frac{16}{9}$,

Consider $\frac{t_2}{t_1} = \boxed{} = \frac{-4}{3}$

$\frac{t_4}{t_3} = \frac{\frac{-64}{27}}{\frac{16}{9}} = \boxed{}$

Here the ratio is constant. Hence the given sequence is a G.P.

Now $t_9 = \boxed{} = ar^8 = 1\left(\frac{-4}{3}\right)^8 = \boxed{}$

Activity 4.2:

For a G.P. $a = 3$, $r = 2$, $S_n = 765$, find n.

Solution : $S_n = 765 = \boxed{}\,(2^n - 1)$,

$\frac{765}{3} = 255 = 2^n - 1$,

$2^n = \boxed{} = 2^8$, n = $\boxed{}$

Activity 4.3:

For a G.P. if $t_6 = 486$, $t_3 = 18$, find t_9

Solution : We know that, for a G.P. $t_n = \boxed{}$

$\therefore\ t_3 = \boxed{} = 18$ ... (I)

$t_6 = \boxed{} = 486$ ... (II)

$\therefore\ \frac{t_6}{t_3} = \frac{\boxed{}}{\boxed{}} = \frac{486}{18}$

$\therefore\ r^{\boxed{}} = \boxed{}$

$\therefore \quad r = \boxed{}$

Now from (I), $t_3 = ar^2 = 18$

$\therefore \quad a = \boxed{}$

$\therefore \quad t_9 = ar^8 = \boxed{}$

Activity 4.4:

If a, b, c, d are in G.P. then show that (a – b) (b – c) and (c – d) are also in G.P.

Solution : a, b, c, d are in G.P.

$\therefore \quad b^2 = \boxed{}$

$\boxed{} = bd$

$ad = \boxed{}$

To prove that (a – b), $(b - c)$, $(c - d)$ are in G.P.

i.e. to prove that $(b - c)^2 = (a - b)(c - d)$

$$
\begin{aligned}
\therefore \quad RHS &= (a - b)(c - d) \\
&= ac - \boxed{} - bc + \boxed{} \\
&= b^2 - bc - bc + \boxed{} \\
&= b^2 - 2bc + c^2 \\
&= \boxed{} \\
&= LHS
\end{aligned}
$$

Activity 4.5:

For a sequence, $S_n = 7(4^n - 1)$, find t_n and show that the sequence is a G.P.

Solution :

$$
\begin{aligned}
S_n &= 7(4^n - 1) \\
S_{(n-1)} &= 7\boxed{} \\
t_n &= \boxed{} \\
&= 7(4^n - 1) - 7(\boxed{}) \\
&= 7[4^n - 1 - 4^{n-1} + 1] \\
&= 7\boxed{}
\end{aligned}
$$

Activity 4.6:

10 people visited an exibition on the first day. The number of visitors was doubled on the next day and so on. Find i) number of visitors on 9^{th} day. ii) Total number of visitors after 12 days.

Solution : On 1^{st} day number of visitors was $\boxed{}$

Number of visitors doubles on next day.

$\therefore$ On 2^{nd} day number of visitors = $\boxed{}$

$\therefore$ On 3^{rd} day number of visitors = $\boxed{}$

and so on

$\therefore$ Number of visitors are 10, 20, 40, 80, ...

These number forms a G.P. with $a = \boxed{}$

$r = \boxed{}$

$\therefore$ No. of visitors on 9^{th} day i.e. $t_9 = a.r^{9-1}$

$$
\begin{aligned}
&= \boxed{}\,2^{\boxed{}} \\
&= 10 \times 2^{\boxed{}} \\
&= 10 \times \boxed{} =
\end{aligned}
$$

Total number of visitors after 12 days

$$
\begin{aligned}
&= S_{12} = a\boxed{} \\
&= 10\left[\frac{1-2^{12}}{1-2}\right] = \frac{10 \times (1\text{-}4096)}{\boxed{}} \\
&= 10 \times 4095 \\
&= \boxed{}
\end{aligned}
$$

Activity 4.7:

Complete the following activity to find sum to n terms of 7+77+777+7777+..............

Let S_n = 7+77+777+7777+........... upto n terms

$= 7\times(\boxed{}+\boxed{}+\boxed{}+\boxed{}$........... upto n terms)

$= \frac{7}{9}(\boxed{}+\boxed{}+\boxed{}+\boxed{}$........... upto n terms)

$= \frac{7}{9}[(10-1)+(\ \ -\ \ )+(\ \ -\ \ )+(\ \ -\ \ )(\ \ -\ \ )$ .... upto n terms)

$$= \frac{7}{9}\,[(10+10^2+\text{..........}+ \text{ upto } n \text{ terms})$$

$$- (1+1+\text{..........upto n terms})]$$

$$= \frac{7}{9}\,[\square(\quad)-\square] \quad \text{.......using} \left(\frac{a.(r^n-1)}{r-1}\right)$$

$$S_n = \frac{7}{9}\left(\frac{\square}{\square}(\quad)-n\right) = \square$$

Activity 4.8:

An empty bus arrived at a bus stand. In the first minute two persons boarded the bus. In the second minute 4 persons, in the third minute 8 persons boarded the bus and so on. The bus was full to its seating capacity in 5 minutes. What was the number of seats in the bus?

Solution : In the first minute, 2 persons board.

In the second minute, 4 persons board.

and so on ....

Hence it is a $\square$

with a = $\square$, $r = \square$

The bus was full in 5 minutes

$$S_5 = a\left(\frac{r^5-1}{r-1}\right) = 2\left(\frac{\square-1}{\square-1}\right)$$

$$= 2\left(\frac{\square}{\square}\right) = \square$$

The number of seats in the bus = $\square$

Let's remember!

For a G.P. $\{t_n\}$,

1) $\dfrac{t_{n+1}}{t_n}$ = constant , $\forall\ n \in N$

2) $t_n = a\,r^{\,n-1}, a \neq 0, r \neq 0$, $\forall\ n \in N$

3) 3 successive terms in G. P. are written as $\frac{a}{r}$, a, ar.

4) 4 successive terms in G. P. are written as $\frac{a}{r^3}$, $\frac{a}{r}$, ar , ar^3. (ratio r^2)

5) 5 successive terms in G. P. are written as $\frac{a}{r^2}$, $\frac{a}{r}$, a , ar , ar^2

EXERCISE 4.1

1) Verify whether the following sequences are G.P. If so, find t_n.

   i) 2,6,18,54,......................

   ii) 1,-5,25,-125...................

   iii) $\sqrt{5}$, $\frac{1}{\sqrt{5}}$, $\frac{1}{5\sqrt{5}}$, $\frac{1}{25\sqrt{5}}$, ...

   iv) 3, 4, 5, 6,.........

   v) 7, 14, 21, 28,.........

2) For the G.P.

   i) if $r = \frac{1}{3}$, $a = 9$; find t_7

   ii) if $a = \frac{7}{243}$, $r = \frac{1}{3}$ find t_3

   iii) if $a = 7$, $r = -3$ find t_6

   iv) if $a = \frac{2}{3}$, $t_6 = 162$, find r

3) Which term of the G.P. 5,25,125,625,.......is 5^{10}?

4) For what values of x.

   $\frac{4}{3}$, x , $\frac{4}{27}$ are in G.P. ?

5) If for a sequence, $t_n = \frac{5^{n-3}}{2^{n-3}}$, show that the sequence is a G.P.

   Find its first term and the common ratio.

6) Find three numbers in G.P. such that their sum is 21 and sum of their squares is 189.

7) Find four numbers in G.P. such that sum of the middle two numbers is 10/3 and their product is 1.

8) Find five numbers in G. P. such that their product is 1024 and fifth term is square of the third term.

9) The fifth term of a G.P. is x , eighth term of the G.P. is y and eleventh term of the G.P. is z Verify whether $y^2 = x\,z$.

10) If p,q,r,s are in G.P. show that $p+q$, $q+r$, $r+s$ are also in G.P.

Let's learn.

4.4 Sum of the first n terms of a G.P.

If $\{t_n\}$is a geometric progression with first term a and common ratio r ; where $a \neq 0$, $r \neq 0$; then the sum of its first n terms is given by

$S_n = a+ar+ar^2+ar^3+ \ldots\ldots\ldots\ldots\ldots\ldots + ar^{n-1} =$

$$a\left(\frac{1-r^n}{1-r}\right), r \neq 1$$

Proof : Consider

$S=1+r+r^2+r^3+\ldots\ldots\ldots r^{n-1}$ ....(i)

Multiplying both sides by r we get

$r\,S = r+r^2+r^3+\ldots\ldots\ldots r^n$ ....(ii)

Subtract (ii) from (i) we get $S - S\,r = 1\text{-}r^n$

$\therefore\ S(1\text{-r}) = 1\text{-r}^n$

$$\therefore\ S = \left(\frac{1-r^n}{1-r}\right), \ldots\ldots\ldots (iii),\quad r \neq 1.$$

Multiplying both sides of equation (i) by a we get ,

$a\,S = a+ar+ar^2+\ldots\ldots\ldots\ldots\ldots + ar^{n-1} = S_n$

$$a\left(\frac{1-r^n}{1-r}\right) = S_n \text{ \{from (i) and (iii)\}}$$

$$S_n = a\left(\frac{1-r^n}{1-r}\right), r \neq 1.$$

If we Subtract (i) from (ii) we get,

$$S_n = a\left(\frac{r^n-1}{r-1}\right).$$

Let's Note.

1. S_n = a. S

2. If r = 1, then $S_n = n\,a$

Solved Examples

Ex 1) If $a = 1$, $r = 2$ find S_n for the G.P.

Solution : $a = 1$, $r = 2$

$$S_n = a\left(\frac{1-r^n}{1-r}\right) = 1\left(\frac{1-2^n}{1-2}\right) = 2^n - 1 .$$

Ex 2) For a G.P. 0.02,0.04,0.08,0.16,…, find S_n.

Solution : Here a = 0.02, r = 2

$$S_n = a\left(\frac{1-r^n}{1-r}\right) = 0.02\left(\frac{1-2^n}{1-2}\right)$$

$= 0.02 \,.\, (2^n \text{-}1)$

Ex 3) For the G.P. 3,–3,3,–3,……… , Find S_n.

Solution :

If n is even , $n = 2k$

$S_{2k} = (3–3) + (3–3) + (3–3) +$ --------- $+ (3–3) = 0$.

If n is odd , $n = 2k+1$

$S_{2k+1} = S_{2k} + t_{2k+1} = 0 + 3 = 3.$

Ex 4) For a G.P. if a=6 , r =2 , find S_{10}.

Solution: $S_n = a\left(\frac{1-r^n}{1-r}\right)$,

$$S_{10} = 6\left(\frac{1-2^{10}}{1-2}\right) = 6\left(\frac{1-1024}{-1}\right)$$

$$= 6\left(\frac{-1023}{-1}\right) = 6\,(1023) = 6138.$$

Ex 5) If for a G.P. r=2 , S_{10}=1023, find *a*.

Solution : $S_{10} = a\left(\frac{1-2^{10}}{1-2}\right)$

$\therefore\ 1023 = a\,(1023)$

$\therefore\ a = 1.$

Ex 6) For a G.P. $a = 5$, $r = 2$, $S_n = 5115$, find n .

Solution : $S_n = 5115 = 5\left(\frac{2^n-1}{2-1}\right) = 5\,(2^n-1)$,

$$\therefore\ \frac{5115}{5} = 1023 = 2^n - 1$$

$$2^n = 1024 = 2^{10}$$

$$\therefore\ n = 10$$

Ex 7) If for a G.P. $S_3 = 16$, $S_6 = 144$, find the first term and the common ratio of the G.P.

Solution : Given

$$S_3 = a\left(\frac{1-r^3}{1-r}\right) = 16 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

$$S_6 = a\left(\frac{1-r^6}{1-r}\right) = 144 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Dividing (2) by (1) we get ,

$$\frac{S_6}{S_3} = \frac{r^6-1}{r^3-1} = \frac{144}{16} = 9$$

$$\frac{(r^3-1)(r^3+1)}{(r^3-1)} = 9,$$

$$(r^3+1) = 9,$$

$$r^3 = 8 = 2^3,$$

$$r = 2\,.$$

Substitute r = 2 in (1) We get

$$a\left(\frac{1-2^3}{1-2}\right) = 16\,,$$

$$a\left(\frac{1-8}{1-2}\right) = 16\,,$$

$$a\,(7) = 16\,,$$

$$a = \frac{16}{7}$$

Ex 8) Find the sum

9+99+999+9999+..................upto n terms.

Solution : Let S_n = 9+99+999+9999+............ ....upto n terms.

S_n = (10-1) + (100-1) + (1000-1)... to n brackets.

= (10+100+1000+ ...... upto n terms)
 – (1+1+1 ...... upto n terms)

Terms in first bracket are in G.P. with a = 10, r = 10 and terms in second bracket are in G.P. with a = r = 1

$$\therefore\ S_n = 10\left(\frac{10^n-1}{10-1}\right) - n$$

$$= \frac{10}{9}\,(10^n - 1) - n\,.$$

Ex 9) Find the sum 5+55+555+5555+........ upto n terms.

Solution: Let S_n

= 5+55+555+5555+............ upto *n* terms.

= 5 (1+11+111+ .... upto *n* terms)

= $\frac{5}{9}$ (9+99+999+ .... upto *n* terms)

= $\frac{5}{9}$ [(10-1) + (100-1) + (1000-1) + ... to *n* brackets]

= $\frac{5}{9}$ [(10 + 100 + 1000 +......... upto *n* terms)

$- (1+1+1+ \dots\dots\dots$ upto n terms)]

$$= \frac{5}{9}\left[10\left(\frac{10^n-1}{10-1}\right) - n\right]$$

$$= \frac{5}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$$

Ex 10) Find the sum to n terms

0.3+0.03+0.003+.......... upto n terms

Solution : Let S_n

= 0.3+0.03+0.003+ upto n terms

= 3 [0.1+ +0.01+ 0.001+ ....upto n terms]

$$= 3\left[\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots\dots\dots + \frac{1}{10^n}\right]$$

$$= 3 \times a\left(\frac{1-r^n}{1-r}\right) \text{ where } a = \frac{1}{10} \text{ and } r = \frac{1}{10}$$

$$= 3 \times \frac{1}{10}\left(\frac{1-\frac{1}{10^n}}{1-\frac{1}{10}}\right)$$

$$= \frac{1}{3}(1 - 0.1^n)$$

Ex 11) Find the n^{th} term of the sequence

0.4 ,0.44,0.444,.........................

Solution : Here t_1=0.4

t_2=0.44 = 0.4 + 0.04

t_3= 0.444 = 0.4+0.04+0.004

....

t_n= 0.4 + 0.04 + 0.004 + 0.0004 + ... upto n terms

here t_n is the sum of first n terms of a G.P. with a = 0.4 and r = 0.1

$$t_n = 0.4\left(\frac{1-0.1^n}{1-0.1}\right) = \frac{4}{9}[1 - (0.1)^n]$$

Ex 12) For a sequence , if S_n = 5(4^n-1) , find the n^{th} term , hence verify that it is a G.P. ,Also find r .

Solution : S_n = 5(4^n–1) , S_{n-1}= 5 (4^{n-1} –1)

We know that $t_n = S_n - S_{n-1}$

$= 5(4^n - 1) - 5 (4^{n-1} - 1)$

$= 5 (4^n) - 5 - 5(4^{n-1}) + 5$

$= 5(4^n - 4^{n-1})$

$= 5 (4^n - 4^n.4^{-1})$

$= 5 (4^n) \left(1 - \frac{1}{4}\right)$

$= 5 (4^n) \left(\frac{3}{4}\right)$

$\therefore t_{n+1} = 5 (4^{n+1}) \times \frac{3}{4}$

Consider $\frac{t_{n+1}}{t_n} = \frac{5(4^{n+1})}{5(4^n)} = 4$ = constant ,

$\forall\ n \in$ N.

$\therefore r = 4$

$\therefore$ the sequence is a G.P.

Ex 13) Which term of the sequence

$\sqrt{3}$, 3, 3$\sqrt{3}$, ... is 243?

Solution : Here a = $\sqrt{3}$, r = $\sqrt{3}$, t_n= 243

$\therefore$ a. r^{n-1} = 243

$$\sqrt{3}\ .\left(\sqrt{3}\right)^{n-1} = 243 = 3^5 = \left(\sqrt{3}\right)^{10}$$

$$\therefore \left(\sqrt{3}\right)^{n} = \left(\sqrt{3}\right)^{10}$$

$\therefore$ n = 10.

Tenth term of the sequence is 243.

Ex 14) How many terms of G.P.

2 , 2^2, 2^3,2^4.,................ are needed to give the sum 2046.

Solution : Here a=2 , r = 2 , let S_n = 2046.

$\therefore\ 2046 = a\left(\frac{r^n-1}{r-1}\right) = 2\left[\frac{2^n-1}{2-1}\right] = 2\,(\,2^n\text{-}1\,)$

$1023 = 2^n\text{-}1\ ,\ 2^n = 1024\ = 2^{10}\ \therefore\ n = 10$

Ex 15) Mr. Pritesh got the job with an annual salary package of Rs. 400000 with 10% annual increment .Find his salary in the 5 th year and also find his total earnings through salary in 10 years.

[Given $(1.1)^4 = 1.4641$, $(1.1)^{10} = 2.59374$

Solution : In the first year he will get a salary of Rs. 400000 .

He gets an increment of 10% so in the second year his salary will be

$$400000 \times \left(\frac{110}{100}\right) = 440000$$

In the third year his salary will be

$400000 \times \left(\frac{110}{100}\right)^2$ and so on ....

Hence it is a G.P. with a = 400000 & r = 1.1.

Similarly his salary in the fifth year will be

$$t_5 = ar^4 = 400000\left(\frac{110}{100}\right)^4 = 585640.$$

$$[\because (1.1)^4 = 1.4641]$$

His total income through salary in 10 years

will be $S_{10} = a\left(\frac{\mathrm{r}^{10} - 1}{\mathrm{r} - 1}\right)$

$= 400000 \times \left(\frac{2.59374 - 1}{0.1}\right)$

$[\because (\,1.1\,)^{10} = 2.59374]$

$= 400000\left(\frac{1.59374}{0.1}\right)$

$= 400000\ [\ 15.9374\ ] =\ 63{,}74{,}960.$

$\therefore$ Mr. Pritesh will get Rs.5,85,640 in the fifth year and his total earnings through salary in 10 years will be Rs. 63,74,960.

Ex 16) A teacher wanted to reward a student by giving some chocolates. He gave the student two choices. He could either have 50 chocolates at once or he could get 1 chocolate on the first day, 2 on the second day, 4 on the third day and so on for 6 days. Which option should the student choose to get more chocolates?

Ans : We need to find sum of chocolates in 6 days.

According to second option teacher gives 1 chocolate on the first day, 2 on the second day, 4 on the third day, and so on. Hence it is a G. P . with $a = 1$, $r = 2$.

If the number of chocolates collected in this way is greater than 50 we have to assume this is the better way.

By using $S_n = a\left(\frac{r^n-1}{r-1}\right)$

$= 1\left(\frac{2^6-1}{2-1}\right)$

$= 64 - 1 = 63$

Hence the student should choose the second way to get more chocolates.

EXERCISE 4.2

1) For the following G.P.s, find S_n

   i) 3 , 6 , 12 , 24 , ...

   ii) $p\ ,q\ ,\frac{q^2}{p}, \frac{q^3}{p^2}$, ...

2) For a G.P. if

   i) $a = 2, r = -\frac{2}{3}$, find S_6

   ii) $S_5 = 1023$, $r = 4$, Find a

3) For a G.P. if

   i) $a = 2$, $r = 3$, $S_n = 242$ find n.

   ii) sum of first 3 terms is 125 and sum of next 3 terms is 27, find the value of r.

4) For a G.P.

i) If $t_3 = 20$, $t_6 = 160$, find S_7

ii) If $t_4 = 16$, $t_9 = 512$, find S_{10}

5) Find the sum to n terms

i) $3 + 33 + 333 + 3333 + \ldots\ldots\ldots\ldots$

ii) $8 + 88 + 888 + 8888 + \ldots\ldots\ldots\ldots$

6) Find the sum to n terms

i) $0.4 + 0.44 + 0.444 + \ldots\ldots\ldots\ldots$

ii) $0.7 + 0.77 + 0.777 + \ldots\ldots\ldots\ldots$

7) Find the n^{th} term of the sequence

i) 0.5, 0.55, 0.555, ...

ii) 0.2, 0.22, 0.222, ...

8) For a sequence, if $S_n = 2(3^n - 1)$, find the n^{th} term , hence show that the sequence is a G.P.

9) If S,P,R are the sum , product and sum of the reciprocals of n terms of a G. P. respectively , then verify that $\left(\frac{S}{R}\right)^n = P^2$.

10) If S_n ,S_{2n} ,S_{3n} are the sum of n,2n,3n terms of a G.P. respectively , then verify that $S_n (S_{3n} - S_{2n}) = (S_{2n} - S_n)^2$.

4.5 Sum of infinite terms of a G. P.

We have learnt how to find the sum of first n terms of a G.P.

If the G.P. is infinite, does it have a finite sum?

Let's understand

Let's find sum to infinity.How can we find it?

We know that for a G.P.

$$S_n = a\left(\frac{1-r^n}{1-r}\right) = \frac{a}{1-r} - \left(\frac{a}{1-r}\right) r^n$$

If $|r| < 1$ then,as n tends to infinity, r^n tends to zero.

Hence S_n tends to $\frac{a}{1-r}$.

(as $\left(\frac{a}{1-r}\right) r^n$ tends to zero)

Hence the sum of an infinite G.P. is given by $\frac{a}{1-r}$, when $|r| < 1$.

Note : If $|r| \geq 1$ then sum to infinite terms does not exist.

SOLVED EXAMPLES

EX 1) Determine whether the sum of all the terms in the series is finite ?

In case it is finite find it.

i) $\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots\ldots\ldots\ldots$

ii) $1, \frac{-1}{2}, \frac{1}{4}, \frac{-1}{8}, \ldots\ldots\ldots\ldots\ldots\ldots$

iii) $-\frac{3}{5}, \frac{-9}{25}, \frac{-27}{125}, \frac{-81}{625} \ldots\ldots\ldots\ldots$

iv) 1,-3,9,-27,81, ...

Solution: i) $r = \frac{1}{3}$

Here $a = \frac{1}{3}$, $r = \frac{1}{3}$, $|r| < 1$

$\therefore$ Sum to infinity exists.

$$S = \frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \left[\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)}\right] = \frac{1}{2}$$

ii) Here $a = 1$, $r = -\frac{1}{2}$, $|r| < 1$

$\therefore$ Sum to infinity exist

$$S = \frac{a}{1-r} = \frac{1}{1-(-\frac{1}{2})} = \frac{1}{\left(\frac{3}{2}\right)} = \frac{2}{3}.$$

iii) Here $a = -\frac{3}{5}$, $r = \frac{3}{5}$, $|r| < 1$

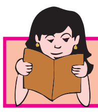
$\therefore$ Sum to infinity exists

$$S = \frac{a}{1-r} = \frac{-\frac{3}{5}}{1-\left(\frac{3}{5}\right)} = \frac{-\left(\frac{3}{5}\right)}{\left(\frac{2}{5}\right)} = \frac{-3}{2}.$$

iv) Here a=1 , r = -3

As $|r| \nless 1$

$\therefore$ Sum to infinity does not exist.



Let's learn.

RECURRING DECIMALS :

We know that every rational number has decimal form.

For example ,

$\frac{7}{6} = 1.166666......... = 1.1\dot{6}$

$\frac{5}{6} = 0.833333.... = 0.8\dot{3}$

$\frac{-5}{3} = -1.666666 = -1.\dot{6}$

$\frac{22}{7} = 3.142857142857....... = 3.\overline{142857}$

$\frac{23}{99} = 0.23232323........... = 0.\overline{23}$

We can use G.P. to represent recurring decimals as a rational number .

SOLVED EXAMPLES

Ex i) 0.66666.....

$= 0.6 + 0.06 + 0.006 + \ldots\ldots\ldots$

$= \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \ldots\ldots ,$

the terms are in G.P. with $a = 0.6$, $r = 0.1 < 1$

$\therefore$ Sum to infinity exists and is given by

$$\frac{a}{1-r} = \frac{0.6}{1-0.1} = \frac{0.6}{0.9} = \frac{6}{9} = \frac{2}{3}$$

ii) $0.\overline{46} = 0.46+0.0046+0.000046 + \ldots\ldots$ the terms are in G.P. with a = 0.46 , $r = 0.01 < 1$.

$\therefore$ Sum to infinity exists

$$= \frac{a}{1-r} = \frac{0.46}{1-(0.01)} = \frac{0.46}{0.99} = \frac{46}{99}.$$

iii) $2.\bar{5} = 2 + 0.5 + 0.05 + 0.005 + 0.0005 + \ldots\ldots$ After the first term,the terms are in G.P. with $a = 0.5$, $r = 0.1 < 1$

$\therefore$ Sum to infinity exists

$$= \frac{a}{1-r} = \frac{0.5}{1-0.1} = \frac{0.5}{0.9} = \frac{5}{9}$$

$\therefore$ $2.\bar{5} = 2 + 0.5 + 0.05 + 0.005 + 0.0005 + \ldots\ldots$

$$= 2 + \frac{5}{9} = \frac{23}{9}$$

EXERCISE 4.3

1) Determine whether the sums to infinity of the following G.P.s exist ,if exist find them

i) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\ldots\ldots\ldots\ldots\ldots$

ii) $2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27} \ldots\ldots$

iii) $-3, 1, \frac{-1}{3}, \frac{1}{9}, \ldots\ldots\ldots\ldots\ldots$

iv) $\frac{1}{5}, \frac{-2}{5}, \frac{4}{5}, \frac{-8}{5}, \frac{16}{5}, \ldots\ldots\ldots\ldots\ldots$

2) Express the following recurring decimals as a rational number.

i) $0.\overline{32}$.

ii) $3.\dot{5}$

iii) $4.\overline{18}$

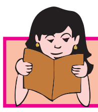
iv) $0.3\overline{45}$

v) $3.4\overline{56}$

3) If the common ratio of a G.P. is $\frac{2}{3}$ and sum of its terms to infinity is 12. Find the first term.

4) If the first term of a G.P. is 16 and sum of its terms to infinity is $\frac{176}{5}$, find the common ratio.

5) The sum of the terms of an infinite G.P. is 5 and the sum of the squares of those terms is 15. Find the G.P.



Let's learn.

Harmonic Progression (H. P.)

Definition : A sequence $t_1, t_2, t_3, t_4, \ldots\ldots\ldots\ldots\ldots t_n,$

($t_n \neq 0$, $n \in$ N) is called a harmonic progression if

$\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}, \ldots\ldots\, \frac{1}{t_n}, \ldots\ldots$ are in A.P.

For example ,

i) $\frac{1}{7}, \frac{1}{11}, \frac{1}{15} \ldots\ldots$

ii) $\frac{1}{4}, \frac{1}{9}, \frac{1}{14}, \frac{1}{19} \ldots\ldots$

iii) $\frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \frac{1}{14} \ldots\ldots$

iv) $\frac{1}{4}, \frac{3}{14}, \frac{3}{16}, \frac{1}{6} \ldots\ldots\ldots$

SOLVED EXAMPLES

Ex1) Find the n^{th} term of the H.P.

$\frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7} \ldots\ldots\ldots$

Solution : Here $2, \frac{5}{2}, 3, \frac{7}{2}$ .... are in A.P. with a = 2 and d = $\frac{1}{2}$ hence

$\frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7}, \ldots\ldots\ldots\ldots$are in H.P.

For A.P.

$t_n = a + (n\text{-}1)\, d = 2 + (n\text{-}1)\, \frac{1}{2}$

$= 2 + \frac{1}{2} n - \frac{1}{2}$

$t_n = \frac{3}{2} + \frac{n}{2} = \frac{3+n}{2}$

For H.P. $t_n = \frac{2}{3+n}$

Ex2) Find the n^{th} term of H.P. $\frac{1}{5}, 1, \frac{-1}{3}, \frac{-1}{7}, \ldots\ldots\ldots\ldots$

Solution: Since 5,1,-3,-7,............... are in A.P.

with $a = 5$ and $d = -4$

Hence $t_n = a + (n-1)\, d$

$= 5 + (n-1)(-4)$

$= 5 - 4n + 4 = 9 - 4n.$

For H.P. $t_n = \frac{1}{9-4n}$

Activity : 4.9

Find the n^{th} term of the following H.P.

$\frac{1}{2}, \frac{1}{7}, \frac{1}{12}, \frac{1}{17}, \frac{1}{22} \ldots\ldots\ldots$

Solution : Here 2, 7, 12, 17, 22 .... are in $\boxed{}$

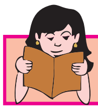
with $a = \boxed{}$ and $(d) = \boxed{}$

hence $\frac{1}{2}, \frac{1}{7}, \frac{1}{12}, \frac{1}{17}, \frac{1}{22}, \ldots\ldots\ldots$ are in $\boxed{}$

$$t_n = a + (n-1)\, d =$$

$$= \boxed{} + (n-1) \boxed{}$$

$$\therefore\ t_n \text{ of the H.P.} = \boxed{}$$



Let's learn.

Types of Means:

Arithmetic mean (A. M.)

If x and y are two numbers, their A.M. is defined by $A = \frac{x+y}{2}$.

We observe that x, A , y form an AP.

Geometric mean (G. M.)

If x and y are two numbers having same sign (positive or negative), their G.M. is defined by

$G = \sqrt{xy}$. We observe that x, G, y form a G P .

Harmonic mean (H. M.)

If x and y are two numbers, their H.M. is defined by $H = \frac{2xy}{x+y}$.

We observe that x, H, y form an HP .

These definitions can be extended to n numbers as follows

$$A = \frac{x_1 + x_2 + x_3 + \ldots\ldots x_n}{n}$$

$$G = \sqrt[n]{x_1 . x_2 . x_3 \ldots\ldots . x_n}$$

$$\text{H.M.} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$$

Theorem : If A, G and H are A.M., G.M., H.M. of two positive numbers respectively, then

i) $G^2 = AH$ ii) $H < G < A$

Proof : let x and y be the two positive numbers.

$$A = \frac{x+y}{2}, \quad G = \sqrt{xy}, \quad H = \frac{2xy}{x+y}$$

$$\text{RHS} = AH = \frac{x+y}{2} \; \frac{2xy}{x+y}$$

$$= xy = G^2 = \text{L.H.S.}$$

$$\text{Consider } A-G = \frac{x+y}{2} - \sqrt{xy}$$

$$= \frac{1}{2}(x+y-2\sqrt{xy})$$

$$A-G = \frac{1}{2}(\sqrt{x} - \sqrt{y})^2 > 0$$

$\therefore$ A > G................................ (I)

$\therefore \frac{A}{G} > 1$................................ (II)

Now consider $G^2 = AH$

$\frac{G}{H} = \frac{A}{G} > 1$ (FROM II)

$\therefore \frac{G}{H} > 1 \therefore G > H$ .... (III)

From (I) and (III) $H < G < A$

Note : If $x = y$ then H = G = A

n arithmetic means between a and b :

Let $A_1, A_2, A_3, \ldots\ldots\ldots$ be the n A.M.s between a and b ,

then $a, A_1, A_2, A_3, \ldots\ldots\ldots, A_n, b$ is an A.P.

Here total number of terms are n+2

$$b = t_{n+2} = a + [(n+2) - 1]\, d$$

$$b = a + (n+1)\, d$$

$$d = \frac{b-a}{n+1}$$

$$A_1 = a + d = a + \frac{b-a}{n+1}$$

$$A_2 = a + 2d = a + 2\,\frac{b-a}{n+1}$$

$$A_3 = a + 3d = a + 3\,\frac{b-a}{n+1}$$

.

.

$$A_n = a + n\,d$$

$$= a + n\,\frac{b-a}{n+1} = \frac{a(n+1)}{n+1} + n\,\frac{b-a}{n+1}$$

$$= \frac{a\,(n+1) + n\,(b-a)}{n+1}$$

$$A_n = \frac{a+nb}{n+1}\,.$$

n geometric means between a and b :

Let $G_1, G_2, G_3, G_4, \ldots\ldots G_n$ be the n G.M.s between a and b ,

then $a, G_1, G_2, G_3, \ldots\ldots G_n, b$ are in G.P

Here total number of terms are $n+2$

$$\therefore \quad t_{n+2} = b = a\,(r)^{n+1}$$

$$\therefore \quad r^{n+1} = \frac{b}{a}$$

$$\therefore \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$G_2 = a\,r^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+2}} \ldots\ldots\ldots\ldots$$

$$G_n = a\,r^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Examples based on means

Ex : 1 Find A.M.,G.M.,H.M. of the numbers 4 and 16

Solution : Let $x = 4$ and $y = 16$

$$A = \frac{x+y}{2} = \frac{20}{2} = 10 \quad \therefore A = 10$$

$$G = \sqrt{xy} = \sqrt{64} = 8\ ,\ G=8.\ ,$$

$$H = \frac{2xy}{x+y} = \frac{2(4)(16)}{4+16} = \frac{128}{20} = \frac{32}{5}.$$

Ex2) Insert 4 arithmetic means between 2 and 22.

Solution:

let A_1, A_2, A_3, A_4 be 4 arithmetic means between 2 and 22

$\therefore$ 2, A_1, A_2, A_3, A_4, 22 are in AP with

$a = 2, \quad t_6=22, \quad n=6$.

$\therefore \quad 22 = 2 + (6-1)d = 2 + 5d$

$20 = 5d\ ,\ d=4$

$A_1 = a+d = 2+4 = 6$,

$A_2 = a+2d = 2+2\times4 = 2+8 = 10$,

$A_3 = a+3d = 2+3\times4 = 2+12 = 14$

$A_4 = a+4d = 2 + 4\times4 = 2+16 = 18$.

$\therefore$ the 4 arithmetic means between 2 and 22 are 6,10,14,18.

Ex: 3 Insert two numbers between $\frac{2}{9}$ and $\frac{1}{12}$ so that the resulting sequence is a H.P.

Solution : let the required numbers be $\frac{1}{H_1}$ and $\frac{1}{H_2}$

$\therefore \quad \frac{2}{9}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{12}$ are in H.P .

$\frac{9}{2}, H_1, H_2$, 12 are in A.P.

$$t_1 = a = \frac{9}{2},\ t_4 = 12 = a+3d = \frac{9}{2} + 3d\,.$$

$$3d = 12 - \frac{9}{2} = \frac{24-9}{2} = \frac{15}{2}$$

$$d = \frac{5}{2}$$

$$t_2 = H_1 = a+d = \frac{9}{2} + \frac{5}{2} = \frac{14}{2} = 7.$$

$$t_3 = H_2 = a+2d = \frac{9}{2} + 2 \times \frac{5}{2} = \frac{19}{2}\,..$$

For resulting sequence $\frac{1}{7}$ and $\frac{2}{19}$ are to be inserted between $\frac{2}{9}$ and $\frac{1}{12}$

Ex: 4 Insert two numbers between 1 and 27 so that the resulting sequence is a G. P.

Solution: Let the required numbers be G_1 and G_2

$\therefore$ 1, G_1, G_2, 27 are in G.P.

$\therefore$ $t_1 = 1$, $t_2 = G_1$, $t_3 = G_2$, $t_4 = 27$

$\therefore$ $a = 1$, $t_4 = ar^3 = 27$

$\therefore$ $r^3 = 27 = 3^3 \therefore r = 3$.

$t_2 = G_1 = ar = 1 \text{ x } 3 = 3$

$t_3 = G_2 = ar^2 = 1(3)^2 = 9$

$\therefore$ 3 and 9 are the two required numbers.

Ex: 5 The A.M. of two numbers exceeds their G.M. by 2 and their H.M. by 18/5 .Find the numbers.

Solution : Given A= G + 2 $\quad \therefore G = A - 2$

Also $A = H + \frac{18}{5} \quad \therefore H = A - \frac{18}{5}$

We know that $G^2 = AH$

$(A-2)^2 = A\left(A - \frac{18}{5}\right)$

$A^2 - 4A + 2^2 = A^2 - \frac{18}{5}A$

$\frac{18}{5}A - 4A = -4$

$-2A = -4 \times 5, \quad \therefore A = 10$

Also $G = A - 2 = 10 - 2 = 8$

$\therefore$ $A = \frac{x+y}{2} = 10$, $x + y = 20$, $y = 20 - x$.
..............................(i)

Now $G = \sqrt{xy} = 8 \quad \therefore xy = 64$

$\therefore$ $x(20 - x) = 64$

$20x - x^2 = 64$

$x^2 - 20x + 64 = 0$

$(x-16)(x-4) = 0$

$x = 16$ or $x = 4$.

$\therefore$ If $x = 16$, then $y = 4 \quad \because y = 20 - x$

$\therefore$ If $x = 4$, then $y = 16$.

The required numbers are 4 and 16.

EXERCISE 4.4

1) Verify whether the following sequences are H.P.

i) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\ldots\ldots\ldots$

ii) $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \ldots\ldots\ldots\ldots$

iii) $\frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \frac{1}{15}, \ldots\ldots\ldots\ldots$

2) Find the n^{th} term and hence find the 8^{th} term of the following H.P.s

i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \ldots\ldots\ldots\ldots$

ii) $\frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots\ldots\ldots\ldots$

iii) $\frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}, \ldots\ldots\ldots\ldots$

3) Find A.M. of two positive numbers whose G.M. and H. M. are 4 and $\frac{16}{5}$

4) Find H.M. of two positive numbers whose A.M. and G.M. are $\frac{15}{2}$ and 6

5) Find G.M. of two positive numbers whose A.M. and H.M. are 75 and 48

6) Insert two numbers between $\frac{1}{7}$ and $\frac{1}{13}$ so that the resulting sequence is a H.P.

7) Insert two numbers between 1 and −27 so that the resulting sequence is a G.P.

8) Find two numbers whose A.M. exceeds their

G.M. by $\frac{1}{2}$ and their H.M. by $\frac{25}{26}$

9) Find two numbers whose A.M. exceeds G.M. by 7 and their H.M. by $\frac{63}{5}$.

Let's remember!

1) For an A.P. $t_n = a + (n-1)\, d$
2) For a G.P. $t_n = a\, r^{\,n-1.}$
3) A.M. of two numbers $A = \frac{x+y}{2}$
4) G.M. of two numbers $G = \sqrt{xy}$
5) H.M. of two numbers $H = \frac{2xy}{x+y}$
6) $G^2 = AH$
7) If $x =$ y then A = G = H
8) If $x \neq$ y then H < G < A.

[where x and y are any two numbers]

4.6 Special Series (sigma Notation)

The symbol $\sum$ (the Greek letter sigma) is used as the summation sign. The sum $a_1+a_2+a_3+a_4+\ldots+a_n$ is expressed as

$\sum_{r=1}^{n} a_r$ (read as sigma a_r , r going from 1 to n)

For example :

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots + x_n$$

$$\sum_{i=1}^{9} x_i = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9$$

$$\sum_{i=3}^{10} x_i = x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}$$

$$\sum_{i=1}^{n} x_i^2 = x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2$$

$$\sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + x_3 y_3 + \ldots + x_n y_n$$

Let's write some important results using $\sum$ notation

Result: 1)

The sum of the first n natural numbers

$$= \sum_{r=1}^{n} r = \frac{n\,(n+1)}{2}$$

Result 2)

The sum of squares of first n natural numbers

$$= \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

Result 3)

The sum of the cubes of the first n natural

numbers $= \sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2$

4.6.1 Properties of Sigma Notation

i) $\sum_{r=1}^{n} kt_r = k \sum_{r=1}^{n} t_r$,

Where k is a non zero constant.

ii) $\sum_{r=1}^{n} (a_r + b_r) = \sum_{r=1}^{n} a_r + \sum_{r=1}^{n} b_r$

iii) $\sum_{r=1}^{n} 1 = n$

iv) $\sum_{r=1}^{n} k = k \sum_{r=1}^{n} 1 = k\, n,$

Where k is a non zero constant.

SOLVED EXAMPLES

Ex 1) Evaluate $\sum_{r=1}^{n} (8r - 7)$

Solution :

$$\sum_{r=1}^{n} (8r - 7) = \sum_{r=1}^{n} 8r - \sum_{r=1}^{n} 7$$

$$= 8 \sum_{r=1}^{n} r - 7 \sum_{r=1}^{n} 1 = 8\left(\frac{n(n+1)}{2}\right) - 7n$$

$= 4\,(n^2+n) - 7n = 4n^2 + 4n - 7n = 4n^2 - 3n.$

Ex 2) Find $\sum_{r=1}^{17} (3r - 5)$

Solution : $\sum_{r=1}^{17} (3r-5) = \sum_{r=1}^{17} 3r - \sum_{r=1}^{17} 5$

$= 3\sum_{r=1}^{17} r - 5 \sum_{r=1}^{17} 1$

$= 3\,\dfrac{17(17+1)}{2} - 5\,(17)$

$= 3 \times 17 \times \dfrac{18}{2} - 85$

$= 3 \times 17 \times 9 - 85$

$= 51 \times 9 - 85$

$= 459 - 85$

$= 374.$

Ex 3) Find $3^2 + 4^2 + 5^2 + \text{------} + 29^2$.

Solution: $3^2 + 4^2 + 5^2 + \text{------} + 29^2$

$= (1^2 + 2^2 + 3^2 + \text{----} + 29^2) - (1^2 + 2^2)$

$= \sum_{r=1}^{29} r^2 - \sum_{r=1}^{2} r^2$

$= \dfrac{29(29+1)(58+1)}{6} - \dfrac{2(2+1)(4+1)}{6}$

$= 29 \times 30 \times \dfrac{59}{6} - 2 \times 3 \times \dfrac{5}{6}$

$= 29 \times 5 \times 59 - 5$

$= 5\,(29 \times 59 - 1) = 5\,(1711 - 1)$

$= 5\,(1710) = 8550$

Ex 4) Find $100^2 - 99^2 + 98^2 - 97^2 + \text{-------} + 2^2 - 1^2$

Solution : $100^2 - 99^2 + 98^2 - 97^2 + \text{-------} + 2^2 - 1^2$

$= (100^2 + 98^2 + 96^2 + \text{-------} + 2^2) -$

$(99^2 + 97^2 + 95^2 + \text{-------} + 1^2)$

$= \sum_{r=1}^{50} (2r)^2 - \sum_{r=1}^{50} (2r-1)^2$

$= \sum_{r=1}^{50} (4r^2 - 4r^2 + 4r - 1)$

$= \sum_{r=1}^{50} (4r - 1)$

$= \sum_{r=1}^{50} 4r - \sum_{r=1}^{50} 1$

$= 4\sum_{r=1}^{50} r - \sum_{r=1}^{50} 1 = 4 \times \dfrac{50(50+1)}{2} - 50$

$= 2 \times 50 \times 51 - 50$

$= 50\,(2 \times 51 - 1)$

$= 50\,(101)$

$= 5050.$

Ex 5) Find $\sum_{r=1}^{n} \dfrac{1^2+2^2+3^2+\text{-----------}+r^2}{r+1}$

Solution : $\sum_{r=1}^{n} \dfrac{1^2+2^2+3^2+\text{-------------}r^2}{r+1}$

$= \sum_{r=1}^{n} \dfrac{r(r+1)(2r+1)}{6(r+1)}$

$= \dfrac{1}{6} \sum_{r=1}^{n} (2r^2 + r)$

$= \dfrac{1}{6}\left(2 \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r\right)$

$= \dfrac{1}{6}\left[2\,\dfrac{n(n+1)(2n+1)}{6} + \dfrac{n(n+1)}{2}\right]$

$= \dfrac{1}{6}\left[\dfrac{n(n+1)(2n+1)}{3} + \dfrac{n(n+1)}{2}\right]$

$= \dfrac{n(n+1)(4n+5)}{36}$.

Ex 6) Find $1 \times 5 + 3 \times 7 + 5 \times 9 + 7 \times 11$ .... + upto n terms .

Solution : Consider first factor of each term.1,3,5,7,------ are in A.P. with a =1, d=2.

$t_r = a + (r-1)d = 1 + (r-1)\,2 = 2r - 1.$

Also the second factors 5,7,9,11,----- are in A.P. with $a = 5$, $d = 2$.

$t_r = 5 + (r-1)\,2 = 5 + 2r - 2 = 2r + 3$

$$\therefore \quad S_n = \sum_{r=1}^{n} (2r-1)(2r+3)$$

$$= \sum_{r=1}^{n} (4r^2 + 4r - 3)$$

$$= 4 \sum_{r=1}^{n} r^2 + 4 \sum_{r=1}^{n} r - \sum_{r=1}^{n} 3$$

$$= 4 \frac{n(n+1)(2n+1)}{6} + 4 \frac{n(n+1)}{2} - 3n$$

$$= n \left[\frac{2(n+1)(2n+1)}{3} \right] + 2n(n+1) - 3n$$

$$= \frac{n}{3} [2(2n^2 + n + 2n + 1) + 6(n+1) - 9]$$

$$= \frac{n}{3} [(4n^2 + 6n + 2) + 6n - 3]$$

$$= \frac{n}{3} (4n^2 + 12n - 1)$$

Ex 7) If $\dfrac{2+4+6+ \text{---} + \textit{upto n terms}}{1+3+5+ \text{---} + \textit{upto n terms}} = \dfrac{20}{19}$,

Find the value of n.

Solution :

Here the terms in the numerator are even numbers hence the general term is $2r$, terms in the denominator are odd numbers hence the general term is $2r - 1$.

$$\therefore \quad \frac{2+4+6+ \text{---} + \textit{upto n terms}}{1+3+5+ \text{---} + \textit{upto n terms}} = \frac{20}{19}$$

$$\frac{\sum_{r=1}^{n} 2r}{\sum_{r=1}^{n} (2r-1)} = \frac{20}{19}$$

$$\frac{2\sum_{r=1}^{n} r}{2\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1} = \frac{20}{19}$$

$$2\frac{n(n+1)}{2} \times 19 = 20 \times 2\frac{n(n+1)}{2} - 20 \times n$$

$n(n+1)\ 19 = 20\, n(n+1) - 20n$

dividing by n we get, $19(n+1) = 20(n+1) - 20$

$19n + 19 = 20n + 20 - 20$

$n = 19.$

EXERCISE 4.5

1) Find the sum $\sum_{r=1}^{n} (r+1)(2r-1)$

2) Find $\sum_{r=1}^{n} (3r^2 - 2r + 1)$

3) Find $\sum_{r=1}^{n} \dfrac{1+2+3+ \text{---} + r}{r}$

4) Find $\sum_{r=1}^{n} \dfrac{1^3+2^3+ \text{---} + r^3}{r(r+1)}$

5) Find the sum $5 \text{ x } 7 + 9 \times 11 + 13 \times 15 + \text{---}$ upto n terms.

6) Find the sum $2^2+4^2+6^2+8^2+ \text{---}$ upto n terms

7) Find $(70^2 - 69^2) + (68^2 - 67^2) + (66^2 - 65^2) + \text{---} + (2^2 - 1^2)$

8) Find the sum $1\times3\times5 + 3\times5\times7 + 5\times7\times9 + \text{---}$ $(2n-1)(2n+1)(2n+3)$

9) Find n, if

$$\frac{1\times2+2\times3+3\times4+4\times5+ \text{---} + \textit{upto n terms}}{1+2+3+4+ \text{---} + \textit{upto n terms}} = \frac{100}{3}.$$

10) If S_1, S_2 and S_3 are the sums of first n natural numbers, their squares and their cubes respectively then show that

$9 S_2^{\,2} = S_3 (1 + 8 S_1)$.

MISCELLANEOUS EXERCISE - 4

1) In a G.P., the fourth term is 48 and the eighth term is 768 . Find the tenth term.

2) For a G.P. $a = \dfrac{4}{3}$ and $t_7 = \dfrac{243}{1024}$, find the value of r.

3) For a sequence , if $t_n = \dfrac{5^{n-2}}{7^{n-3}}$, verify whether the sequence is a G.P. If it is a G.P. , find its first term and the common ratio.

4) Find three numbers in G.P. such that their sum is 35 and their product is 1000.

5) Find 4 numbers in G.P. such that the sum of middle 2 numbers is 10/3 and their product is 1.

6) Find five numbers in G.P. such that their product is 243 and sum of second and fourth number is 10.

7) For a sequence $S_n = 4(7^n - 1)$ verify whether the sequence is a G.P.

8) Find 2+22+222+2222+ ----------------- upto n terms.

9) Find the n^{th} term of the sequence 0.6,0.66,0.666,0.6666,----------

10) Find $\sum_{r=1}^{n} (5r^2 + 4r - 3)$

11) Find $\sum_{r=1}^{n} r(r-3)(r-2)$

12) Find $\sum_{r=1}^{n} \dfrac{1^2 + 2^2 + 3^2 + \text{--------} + r^2}{2r+1}$

13) Find $\sum_{r=1}^{n} \dfrac{1^3 + 2^3 + 3^3 + \text{-------} + r^3}{(r+1)^2}$

14) Find 2 × 6 + 4 × 9 + 6 × 12 + ----------- upto n terms.

15) Find $12^2 + 13^2 + 14^2 + 15^2 +$ ----------- 20^2

16) Find $(50^2 - 49^2) + (48^2 - 47^2) + (46^2 - 45^2) +$ ----- $+ (2^2 - 1^2)$.

17) In a G.P. if $t_2 = 7$, $t_4 = 1575$ find r

18) Find k so that k-1,k , k+2 are consecutive terms of a G.P.

19) If p^{th} , q^{th} and r^{th} terms of a G.P. are x,y,z respectively, find the value of $x^{q-r} \times y^{r-p} \times z^{p-q}$

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