1. DIFFERENTIATION

Let us Study

- Derivatives of Composite functions.
- Derivatives of Inverse functions
- Derivatives of Implicit functions.
- Higher order Derivatives.
- Geometrical meaning of Derivative.
- Logarithmic Differentiation
- Derivatives of Parametric functions.

Let us Recall

- The derivative of $f(x)$ with respect to x, at $x = a$ is given by $f'(a) = \lim_{h \to 0}\left[\frac{f(a+h) - f(a)}{h}\right]$
- The derivative can also be defined for $f(x)$ at any point x on the open interval as $f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$. If the function is given as $y = f(x)$ then its derivative is written as $\frac{dy}{dx} = f'(x)$.
- For a differentiable function $y = f(x)$ if δx is a small increment in x and the corresponding increment in y is δy then $\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{dy}{dx}$.
- Derivatives of some standard functions.

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
c (Constant)	0
x^n	nx^{n-1}
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\frac{1}{x^n}$	$-\frac{n}{x^{n+1}}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
e^x	e^x
a^x	$a^x \log a$
$\log x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x \log a}$

Table 1.1.1

Rules of Differentiation :

If u and v are differentiable functions of x such that

(i) $y = u \pm v$ then $\dfrac{dy}{dx} = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$ (ii) $y = uv$ then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

(iii) $y = \dfrac{u}{v}$ where $v \neq 0$ then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Introduction :

The history of mathematics presents the development of calculus as being accredited to Sir Isaac Newton (1642-1727) an English physicist and mathematician and Gottfried Wilhelm Leibnitz (1646-1716) a German physicist and mathematician. The Derivative is one of the fundamental ideas of calculus. It's all about rate of change in a function. We try to find interpretations of these changes in a mathematical way. The symbol δ will be used to represent the change, for example δx represents a small change in the variable x and it is read as "change in x" or "increment in x". δy is the corresponding change in y if y is a function of x.

We have already studied the basic concept, derivatives of standard functions and rules of differentiation in previous standard. This year, in this chapter we are going to study the geometrical meaning of derivative, derivatives of Composite, Inverse, Logarithmic, Implicit and Parametric functions and also higher order derivatives. We also add some more rules of differentiation.

Let us Learn

1.1.1 Derivatives of Composite Functions (Function of another function) :

So far we have studied the derivatives of simple functions like $\sin x$, $\log x$, e^x etc. But how about the derivatives of $\sin\sqrt{x}$, $\log\left(\sin(x^2 + 5)\right)$ or $e^{\tan x}$ etc ? These are known as composite functions. In this section let us study how to differentiate composite functions.

1.1.2 Theorem : If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x such that the composite function $y = f[g(x)]$ is a differentiable function of x then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$.

Proof : Given that $y = f(u)$ and $u = g(x)$. We assume that u is not a constant function. Let there be a small increment in the value of x say δx then δu and δy are the corresponding increments in u and y respectively.

As δx, δu, δy are small increments in x, u and y respectively such that $\delta x \neq 0$, $\delta u \neq 0$ and $\delta y \neq 0$.

We have $\dfrac{\delta y}{\delta x} = \dfrac{\delta y}{\delta u} \times \dfrac{\delta u}{\delta x}$.

Taking the limit as $\delta x \to 0$ on both sides we get,

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta u}\right) \times \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right)$$

As $\delta x \to 0$, we get, $\delta u \to 0$ ($\because$ u is a continuous function of x)

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta u \to 0}\left(\frac{\delta y}{\delta u}\right) \times \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right) \qquad \ldots\ldots \text{(I)}$$

Since y is a differentiable function of u and u is a differentiable function of x.
we have,

$$\lim_{\delta u \to 0}\left(\frac{\delta y}{\delta u}\right) = \frac{dy}{du} \quad \text{and} \quad \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right) = \frac{du}{dx} \qquad \ldots\ldots \text{(II)}$$

From (I) and (II), we get

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{dy}{du} \times \frac{du}{dx} \qquad \ldots\ldots \text{(III)}$$

The R.H.S. of (III) exists and is finite, implies L.H.S.of (III) also exists and is finite

$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{dy}{dx}$. Then equation (III) becomes,

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

Note:

1. The derivative of a composite function can also be expressed as follows. $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x such that the composite function $y = f[g(x)]$ is defined then

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x).$$

2. If $y = f(v)$ is a differentiable function of v and $v = g(u)$ is a differentiable function of u and $u = h(x)$ is a differentiable function of x then

$$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dx}.$$

3. If y is a differentiable function of u_1, u_i is a differentiable function of u_{i+1} for $i = 1, 2, \ldots, n-1$ and u_n is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du_1} \times \frac{du_1}{du_2} \times \frac{du_2}{du_3} \times \ldots \times \frac{du_{n-1}}{du_n} \times \frac{du_n}{dx}$$

This rule is also known as **Chain rule**.

1.1.3 Derivatives of some standard Composite Functions :

y	$\frac{dy}{dx}$
$[f(x)]^n$	$n[f(x)]^{n-1} \cdot f'(x)$
$\sqrt{f(x)}$	$\frac{f'(x)}{2\sqrt{f(x)}}$
$\frac{1}{[f(x)]^n}$	$-\frac{n \cdot f'(x)}{[f(x)]^{n+1}}$
$\sin[f(x)]$	$\cos[f(x)] \cdot f'(x)$
$\cos[f(x)]$	$-\sin[f(x)] \cdot f'(x)$
$\tan[f(x)]$	$\sec^2[f(x)] \cdot f'(x)$
$\sec[f(x)]$	$\sec[f(x)] \cdot \tan[f(x)] \cdot f'(x)$

y	$\frac{dy}{dx}$
$\cot[f(x)]$	$-\text{cosec}^2[f(x)] \cdot f'(x)$
$\text{cosec}[f(x)]$	$-\text{cosec}[f(x)] \cdot \cot[f(x)] \cdot f'(x)$
$a^{f(x)}$	$a^{f(x)} \cdot \log a \cdot f'(x)$
$e^{f(x)}$	$e^{f(x)} \cdot f'(x)$
$\log[f(x)]$	$\frac{f'(x)}{f(x)}$
$\log_a[f(x)]$	$\frac{f'(x)}{f(x)\log a}$

Table 1.1.2

SOLVED EXAMPLES

Ex. 1 : Differentiate the following *w. r. t. x*.

(i) $y = \sqrt{x^2 + 5}$ (ii) $y = \sin(\log x)$ (iii) $y = e^{\tan x}$

(iv) $\log(x^5 + 4)$ (v) $5^{3\cos x - 2}$ (vi) $y = \frac{3}{(2x^2 - 7)^5}$

Solution : (i) $y = \sqrt{x^2 + 5}$

Method 1 :

Let $u = x^2 + 5$ then $y = \sqrt{u}$, where y is a differentiable function of u and u is a differentiable function of x then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \quad \ldots\ldots (I)$$

Now, $y = \sqrt{u}$

Differentiate *w. r. t. u*

$$\frac{dy}{du} = \frac{d}{du}(\sqrt{u}) = \frac{1}{2\sqrt{u}} \text{ and } u = x^2 + 5$$

Differentiate *w. r. t. x*

$$\frac{du}{dx} = \frac{d}{dx}(x^2 + 5) = 2x$$

Now, equation (I) becomes,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \times 2x = \frac{x}{\sqrt{x^2 + 5}}$$

Method 2 :

We have $y = \sqrt{x^2 + 5}$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x^2 + 5}\right)$$

[Treat $x^2 + 5$ as u in mind and use the formula of derivative of $\sqrt{u}$]

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2 + 5}} \cdot \frac{d}{dx}(x^2 + 5)$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2 + 5}}(2x)$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 5}}$$

(ii) $y = \sin(\log x)$

Method 1 :

Let $u = \log x$ then $y = \sin u$, where y is a differentiable function of u and u is a differentiable function of x then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad \ldots\ldots (I)$$

Now, $y = \sin u$

Differentiate *w. r. t.* u

$$\frac{dy}{du} = \frac{d}{du}(\sin u) = \cos u \text{ and } u = \log x$$

Differentiate *w. r. t.* x

$$\frac{du}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$

Now, equation (I) becomes,

$$\frac{dy}{dx} = \cos u \times \frac{1}{x} = \frac{\cos(\log x)}{x}$$

Method 2 :

We have $y = \sin(\log x)$

Differentiate *w. r. t.* x

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(\log x)]$$

[Treat $\log x$ as u in mind and use the formula of derivative of $\sin u$]

$$\frac{dy}{dx} = \cos(\log x) \times \frac{d}{dx}(\log x)$$

$$\frac{dy}{dx} = \cos(\log x) \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{\cos(\log x)}{x}$$

Note : Hence onwards let's use Method 2.

(iii) $y = e^{\tan x}$

Differentiate *w. r. t.* x

$$\frac{dy}{dx} = \frac{d}{dx}[e^{\tan x}]$$

$$\frac{dy}{dx} = e^{\tan x} \times \frac{d}{dx}(\tan x)$$

$$\frac{dy}{dx} = e^{\tan x} \cdot \sec^2 x = \sec^2 x \cdot e^{\tan x}$$

(iv) Let $y = \log(x^5 + 4)$

Differentiate *w. r. t.* x

$$\frac{dy}{dx} = \frac{d}{dx}[\log(x^5 + 4)]$$

$$\frac{dy}{dx} = \frac{1}{x^5 + 4} \times \frac{d}{dx}(x^5 + 4)$$

$$\frac{dy}{dx} = \frac{1}{x^5 + 4}(5x^4) = \frac{5x^4}{x^5 + 4}$$

(v) Let $y = 5^{3\cos x - 2}$

Differentiate *w. r. t.* x

$$\frac{dy}{dx} = \frac{d}{dx}[5^{3\cos x - 2}]$$

$$\frac{dy}{dx} = 5^{3\cos x - 2} \cdot \log 5 \times \frac{d}{dx}(3\cos x - 2)$$

$$\frac{dy}{dx} = -3\sin x \cdot 5^{3\cos x - 2} \cdot \log 5$$

(vi) Let $y = \dfrac{3}{(2x^2 - 7)^5}$

Differentiate *w. r. t.* x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3}{(2x^2 - 7)^5}\right) = 3\frac{d}{dx}\left(\frac{1}{(2x^2 - 7)^5}\right)$$

$$= 3 \times \frac{-5}{(2x^2 - 7)^6} \times \frac{d}{dx}(2x^2 - 7)$$

$$= -\frac{15}{(2x^2 - 7)^6}(4x)$$

$$\frac{dy}{dx} = -\frac{60x}{(2x^2 - 7)^6}$$

Ex. 2 : Differentiate the following *w. r. t. x.*

(i) $y = \sqrt{\sin x^3}$ (ii) $y = \cot^2(x^3)$ (iii) $y = \log[\cos(x^5)]$

(iv) $y = (x^3 + 2x - 3)^4 (x + \cos x)^3$ (v) $y = (1 + \cos^2 x)^4 \times \sqrt{x + \sqrt{\tan x}}$

Solution :

(i) $y = \sqrt{\sin x^3}$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\sin x^3}\right)$$

$$= \frac{1}{2\sqrt{\sin x^3}} \times \frac{d}{dx}(\sin x^3)$$

$$= \frac{1}{2\sqrt{\sin x^3}} \times \cos x^3 \times \frac{d}{dx}(x^3)$$

$$= \frac{1}{2\sqrt{\sin x^3}} \times \cos x^3 \times (3x^2)$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 \cos x^3}{2\sqrt{\sin x^3}}$$

(ii) $y = \cot^2(x^3)$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cot^2(x^3)\right)$$

$$= \frac{d}{dx}[\cot(x^3)]^2$$

$$= 2\cot(x^3)\frac{d}{dx}[\cot(x^3)]$$

$$= 2\cot(x^3)[-\operatorname{cosec}^2(x^3)]\frac{d}{dx}(x^3)$$

$$= -2\cot(x^3)\operatorname{cosec}^2(x^3)(3x^2)$$

$$\therefore \frac{dy}{dx} = -6x^2\cot(x^3)\operatorname{cosec}^2(x^3)$$

(iii) $y = \log[\cos(x^5)]$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\log[\cos(x^5)]\right)$$

$$= \frac{1}{\cos(x^5)} \cdot \frac{d}{dx}[\cos(x^5)]$$

$$= \frac{1}{\cos(x^5)}\left(-\sin(x^5)\right)\frac{d}{dx}(x^5)$$

$$\therefore \frac{dy}{dx} = -\tan(x^5)(5x^4) = -5x^4\tan(x^5)$$

(iv) $y = (x^3 + 2x - 3)^4 (x + \cos x)^3$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}\left[(x^3 + 2x - 3)^4 (x + \cos x)^3\right]$$

$$= (x^3 + 2x - 3)^4 \cdot \frac{d}{dx}(x + \cos x)^3 + (x + \cos x)^3 \cdot \frac{d}{dx}(x^3 + 2x - 3)^4$$

$$= (x^3 + 2x - 3)^4 \cdot 3\,(x + \cos x)^2 \cdot \frac{d}{dx}(x + \cos x) + (x + \cos x)^3 \cdot 4(x^3 + 2x - 3)^3 \cdot \frac{d}{dx}(x^3 + 2x - 3)$$

$$= (x^3 + 2x - 3)^4 \cdot 3\,(x + \cos x)^2\,(1 - \sin x) + (x + \cos x)^3 \cdot 4(x^3 + 2x - 3)^3 (3x^2 + 2)$$

$$\therefore \quad \frac{dy}{dx} = 3(x^3 + 2x - 3)^4 (x + \cos x)^2\,(1 - \sin x) + 4\,(3x^2 + 2)\,(x^3 + 2x - 3)^3 (x + \cos x)^3$$

(v) $y = (1 + \cos^2 x)^4 \times \sqrt{x + \sqrt{\tan x}}$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}\left[(1+\cos^2 x)^4 \times \sqrt{x+\sqrt{\tan x}}\right]$$

$$= (1+\cos^2 x)^4 \frac{d}{dx}\left(\sqrt{x+\sqrt{\tan x}}\right) + \left(\sqrt{x+\sqrt{\tan x}}\right)\frac{d}{dx}(1+\cos^2 x)^4$$

$$= (1+\cos^2 x)^4 \cdot \frac{1}{2\sqrt{x+\sqrt{\tan x}}} \cdot \frac{d}{dx}\left(x+\sqrt{\tan x}\right) + \left(\sqrt{x+\sqrt{\tan x}}\right) \cdot 4(1+\cos^2 x)^3 \frac{d}{dx}[1+(\cos x)^2]$$

$$= (1+\cos^2 x)^4 \cdot \frac{1}{2\sqrt{x+\sqrt{\tan x}}}\left(1+\frac{1}{2\sqrt{\tan x}} \cdot \frac{d}{dx}(\tan x)\right) + \left(\sqrt{x+\sqrt{\tan x}}\right) \cdot 4(1+\cos^2 x)^3 (2\cos x)\frac{d}{dx}(\cos x)$$

$$= (1+\cos^2 x)^4 \cdot \frac{1}{2\sqrt{x+\sqrt{\tan x}}}\left(1+\frac{\sec^2 x}{2\sqrt{\tan x}}\right) + \left(\sqrt{x+\sqrt{\tan x}}\right) \cdot 4(1+\cos^2 x)^3 (2\cos x)(-\sin x)$$

$$= (1+\cos^2 x)^4 \cdot \frac{1}{2\sqrt{x+\sqrt{\tan x}}}\left(\frac{2\sqrt{\tan x}+\sec^2 x}{2\sqrt{\tan x}}\right) - \left(\sqrt{x+\sqrt{\tan x}}\right) \cdot 4(1+\cos^2 x)^3 (2\sin x\cos x)$$

$$\frac{dy}{dx} = \frac{(1+\cos^2 x)^4 (2\sqrt{\tan x}+\sec^2 x)}{4\sqrt{\tan x}\sqrt{x+\sqrt{\tan x}}} - 4\sin 2x(1+\cos^2 x)^3\sqrt{x+\sqrt{\tan x}}$$

Ex. 3 : Differentiate the following *w. r. t. x.*

(i) $y = \log_3 (\log_5 x)$

(ii) $y = \log\left[e^{3x} \cdot \frac{(3x-4)^{\frac{2}{3}}}{\sqrt[3]{2x+5}}\right]$

(iii) $y = \log\left[\sqrt{\frac{1-\cos\left(\frac{3x}{2}\right)}{1+\cos\left(\frac{3x}{2}\right)}}\right]$

(iv) $y = \log\left[\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}-x}\right]$

(v) $y = (4)^{\log_2(\sin x)} + (9)^{\log_3(\cos x)}$

(vi) $y = a^{a^{\log_a(\cot x)}}$

Solution :

(i) $y = \log_3 (\log_5 x)$

$$= \log_3 \left(\frac{\log x}{\log 5}\right) \quad = \log_3 (\log x) - \log_3 (\log 5)$$

$$\therefore \quad y = \frac{\log(\log x)}{\log 3} - \log_3 (\log 5)$$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\log(\log x)}{\log 3} - \log_3(\log 5)\right]$$

$$= \frac{1}{\log 3}\frac{d}{dx}[\log(\log x)] - \frac{d}{dx}[\log_3(\log 5)]$$

$$= \frac{1}{\log 3} \times \frac{1}{\log x}\frac{d}{dx}(\log x) - 0 \qquad \text{[Note that } \log_3(\log 5) \text{ is constant]}$$

$$= \frac{1}{\log 3} \times \frac{1}{\log x} \times \frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x \log x \log 3}$$

(ii) $y = \log\left[e^{3x} \cdot \frac{(3x-4)^{\frac{2}{3}}}{\sqrt[3]{2x+5}}\right] = \log\left[\frac{e^{3x} \cdot (3x-4)^{\frac{2}{3}}}{(2x+5)^{\frac{1}{3}}}\right]$

$$= \log\left[e^{3x} \cdot (3x-4)^{\frac{2}{3}}\right] - \log\left[(2x+5)^{\frac{1}{3}}\right]$$

$$= \log e^{3x} + \log(3x-4)^{\frac{2}{3}} - \log(2x+5)^{\frac{1}{3}}$$

$$\therefore \quad y = 3x + \frac{2}{3}\log(3x-4) - \frac{1}{3}\log(2x+5) \qquad [\because \log e = 1]$$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}\left[3x + \frac{2}{3}\log(3x-4) - \frac{1}{3}\log(2x+5)\right]$$

$$= 3\frac{d}{dx}(x) + \frac{2}{3} \cdot \frac{d}{dx}[\log(3x-4)] - \frac{1}{3} \cdot \frac{d}{dx}[\log(2x+5)]$$

$$= 3(1) + \frac{2}{3} \cdot \frac{1}{3x-4} \cdot \frac{d}{dx}(3x-4) - \frac{1}{3} \cdot \frac{1}{2x+5} \cdot \frac{d}{dx}(2x+5)$$

$$= 3(1) + \frac{2}{3} \cdot \frac{1}{3x-4} \cdot (3) - \frac{1}{3} \cdot \frac{1}{2x+5} \cdot (2)$$

$$\therefore \quad \frac{dy}{dx} = 3 + \frac{2}{3x-4} - \frac{2}{3(2x+5)}$$

(iii) $y=\log\left[\sqrt{\dfrac{1-\cos\left(\frac{3x}{2}\right)}{1+\cos\left(\frac{3x}{2}\right)}}\right]=\log\left[\sqrt{\dfrac{2\sin^2\left(\frac{3x}{4}\right)}{2\cos^2\left(\frac{3x}{4}\right)}}\right]$

$\therefore \quad y=\log\left[\tan\left(\frac{3x}{4}\right)\right]$

Differentiate *w. r. t. x*

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\log\left[\tan\left(\frac{3x}{4}\right)\right]\right\}$$

$$=\frac{1}{\tan\left(\frac{3x}{4}\right)}\cdot\frac{d}{dx}\left[\tan\left(\frac{3x}{4}\right)\right]$$

$$=\cot\left(\frac{3x}{4}\right)\cdot\sec^2\left(\frac{3x}{4}\right)\cdot\frac{d}{dx}\left(\frac{3x}{4}\right)$$

$$=\frac{\cos\left(\frac{3x}{4}\right)}{\sin\left(\frac{3x}{4}\right)}\times\frac{1}{\cos^2\left(\frac{3x}{4}\right)}\times\frac{3}{4}$$

$$=\frac{3}{2\left[2\sin\left(\frac{3x}{4}\right)\cdot\cos\left(\frac{3x}{4}\right)\right]}=\frac{3}{2\sin\left(\frac{3x}{2}\right)}$$

$$\therefore \quad \frac{dy}{dx}=\frac{3}{2}\operatorname{cosec}\left(\frac{3x}{2}\right)$$

(iv) $y=\log\left[\dfrac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}-x}\right]=\log\left[\dfrac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}\times\dfrac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}+x}\right]$

$$=\log\left[\frac{\left(\sqrt{x^2+a^2}+x\right)^2}{x^2+a^2-x^2}\right]$$

$$=\log\left[\frac{\left(\sqrt{x^2+a^2}+x\right)^2}{a^2}\right]$$

$$=\log\left(\sqrt{x^2+a^2}+x\right)^2-\log(a^2)$$

$$\therefore \quad y=2\log\left(\sqrt{x^2+a^2}+x\right)-\log(a^2)$$

Differentiate *w. r. t. x*

$$\frac{dy}{dx}=\frac{d}{dx}\left[2\log\left(\sqrt{x^2+a^2}+x\right)-\log(a^2)\right]$$

$$=2\frac{d}{dx}\left[\log\left(\sqrt{x^2+a^2}+x\right)\right]-\frac{d}{dx}[\log(a^2)]$$

$$=2\times\frac{1}{\sqrt{x^2+a^2}+x}\cdot\frac{d}{dx}\left[\sqrt{x^2+a^2}+x\right]-0$$

$$=\frac{2}{\sqrt{x^2+a^2}+x}\cdot\left[\frac{1}{2\sqrt{x^2+a^2}}\cdot\frac{d}{dx}(x^2+a^2)+1\right]$$

$$=\frac{2}{\sqrt{x^2+a^2}+x}\cdot\left[\frac{1}{2\sqrt{x^2+a^2}}(2x)+1\right]$$

$$=\frac{2}{\sqrt{x^2+a^2}+x}\cdot\left[\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}\right]$$

$$\frac{dy}{dx}=\frac{2}{\sqrt{x^2+a^2}}$$

(v) $y=(4)^{\log_2(\sin x)}+(9)^{\log_3(\cos x)}$

$=\left(2^2\right)^{\log_2(\sin x)}+\left(3^2\right)^{\log_3(\cos x)}$

$=\left(2\right)^{2\log_2(\sin x)}+\left(3\right)^{2\log_3(\cos x)}$

$=\left(2\right)^{\log_2(\sin^2 x)}+\left(3\right)^{\log_3(\cos^2 x)}$ $[\because a^{\log_a f(x)}=f(x)]$

$=\sin^2 x+\cos^2 x$

$\therefore\ y=1$

Differentiate *w. r. t. x*

$$\frac{dy}{dx}=\frac{d}{dx}(1)=0$$

(vi) $y=a^{a^{\log_a(\cot x)}}$

$y=a^{\cot x}$ $[\because a^{\log_a f(x)}=f(x)]$

Differentiate *w. r. t. x*

$$\frac{dy}{dx}=\frac{d}{dx}(a^{\cot x})$$

$$=a^{\cot x}\log a\cdot\frac{d}{dx}(\cot x)$$

$$=a^{\cot x}\log a\,(-\operatorname{cosec}^2 x)$$

$$\frac{dy}{dx}=-\operatorname{cosec}^2 x\cdot a^{\cot x}\log a$$

Ex. 4 : If $f(x)=\sqrt{7g(x)-3}$, $g(3)=4$ and $g'(3)=5$, find $f'(3)$.

Solution : Given that : $f(x)=\sqrt{7g(x)-3}$

Differentiate *w. r. t. x*

$$f'(x)=\frac{d}{dx}\left(\sqrt{7g(x)-3}\right)=\frac{1}{2\sqrt{7g(x)-3}}\frac{d}{dx}[7g(x)-3]$$

$$\therefore\ f'(x)=\frac{7g'(x)}{2\sqrt{7g(x)-3}}$$

For $x=3$, we get

$$f'(3)=\frac{7g'(3)}{2\sqrt{7g(3)-3}}=\frac{35}{2(5)}=\frac{7}{2}$$ [Since $g(3)=4$ and $g'(3)=5$]

Ex. 5 : If $F(x) = G\{3G[5G(x)]\}$, $G(0) = 0$ and $G'(0) = 3$, find $F'(0)$.

Solution : Given that : $F(x) = G\{3G[5G(x)]\}$

Differentiate $w.\ r.\ t.\ x$

$$F'(x) = \frac{d}{dx} G\{3G[5G(x)]\}$$
$$= G'\{3G[5G(x)]\} 3 \cdot \frac{d}{dx}[G[5G(x)]]$$
$$= G'\{3G[5G(x)]\} 3 \cdot G'[5G(x)]\, 5 \cdot \frac{d}{dx}[G(x)]$$

$$F'(x) = 15 \cdot G'\{3G[5G(x)]\} G'[5G(x)]\, G'(x)$$

For $x = 0$, we get

$$F'(0) = 15 \cdot G'\{3G[5G(0)]\} G'[5G(0)]\, G'(0)$$
$$= 15 \cdot G'[3G(0)] G'(0) \cdot (3) \qquad [\because G(0) = 0 \text{ and } G'(0) = 3]$$
$$= 15 \cdot G'(0)(3)(3) = 15 \cdot (3)(3)(3) = 405$$

Ex. 6 : Select the appropriate hint from the hint basket and fill in the blank spaces in the following paragraph. [Activity]

"Let $f(x) = \sin x$ and $g(x) = \log x$ then $f[g(x)] =$ _ _ _ _ _ _ _ _ _ _ _ _ _ _ and $g[f(x)] =$ _ _ _ _ _ _ _ _ _ . Now $f'(x) =$ _ _ _ _ _ _ _ _ _ _ _ and $g'(x) =$ _ _ _ _ _ _ _ _ _.

The derivative of $f[g(x)]$ $w.\ r.\ t.\ x$ in terms of f and g is _ _ _ _ _ _ _ _ _ _ _ _ _.

Therefore $\frac{d}{dx}[f[g(x)]] =$ _ _ _ _ _ _ _ _ _ _ and $\left[\frac{d}{dx}[f[g(x)]]\right]_{x=1} =$ _ _ _ _ _ _ _ _ _ _ _ _ _.

The derivative of $g[f(x)]$ $w.\ r.\ t.\ x$ in terms of f and g is _ _ _ _ _ _ _ _ _ _ _ _ _.

Therefore $\frac{d}{dx}[g[f(x)]] =$ _ _ _ _ _ _ _ _ _ _ and $\left[\frac{d}{dx}[g[f(x)]]\right]_{x=\frac{\pi}{3}} =$ _ _ _ _ _ _ _ _ _ _ _ _."

Hint basket : $\left\{ f'[g(x)] \cdot g'(x), \frac{\cos(\log x)}{x}, 1, g'[f(x)] \cdot f'(x), \cot x, \sqrt{3}, \right.$
$\left. \sin(\log x), \log(\sin x), \cos x, \frac{1}{x} \right\}$

Solution : $\sin(\log x)$, $\log(\sin x)$, $\cos x$, $\frac{1}{x}$, $f'[g(x)] \cdot g'(x)$, $\frac{\cos(\log x)}{x}$, 1, $g'[f(x)] \cdot f'(x)$, $\cot x$, $\sqrt{3}$.

EXERCISE 1.1

(1) Differentiate $w.\ r.\ t.\ x$.

(i) $(x^3 - 2x - 1)^5$

(ii) $\left(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5\right)^{\frac{5}{2}}$

(iii) $\sqrt{x^2 + 4x - 7}$

(iv) $\sqrt{x^2 + \sqrt{x^2 + 1}}$

(v) $\dfrac{3}{5\sqrt[3]{(2x^2 - 7x - 5)^5}}$

(vi) $\left(\sqrt{3x - 5} - \dfrac{1}{\sqrt{3x - 5}}\right)^5$

(2) Differentiate the following *w.r.t.* x

(i) $\cos(x^2 + a^2)$ (ii) $\sqrt{e^{(3x+2)}+5}$

(iii) $\log\left[\tan\left(\frac{x}{2}\right)\right]$ (iv) $\sqrt{\tan\sqrt{x}}$

(v) $\cot^3[\log(x^3)]$ (vi) $5^{\sin^3 x + 3}$

(vii) $\operatorname{cosec}(\sqrt{\cos x})$ (viii) $\log[\cos(x^3 - 5)]$

(ix) $e^{3\sin^2 x - 2\cos^2 x}$ (x) $\cos^2[\log(x^2 + 7)]$

(xi) $\tan[\cos(\sin x)]$ (xii) $\sec[\tan(x^4 + 4)]$

(xiii) $e^{\log[(\log x)^2 - \log x^2]}$ (xiv) $\sin\sqrt{\sin\sqrt{x}}$

(xv) $\log[\sec(e^{x^2})]$ (xvi) $\log_{e^2}(\log x)$

(xvii) $[\log[\log(\log x)]]^2$

(xviii) $\sin^2 x^2 - \cos^2 x^2$

(3) Differentiate the following *w.r.t.* x

(i) $(x^2 + 4x + 1)^3 + (x^3 - 5x - 2)^4$

(ii) $(1 + 4x)^5 (3 + x - x^2)^8$

(iii) $\frac{x}{\sqrt{7-3x}}$ (iv) $\frac{(x^3-5)^5}{(x^3+3)^3}$

(v) $(1 + \sin^2 x)^2 (1 + \cos^2 x)^3$

(vi) $\sqrt{\cos x} + \sqrt{\cos\sqrt{x}}$

(vii) $\log(\sec 3x + \tan 3x)$ (viii) $\frac{1+\sin x^\circ}{1-\sin x^\circ}$

(ix) $\cot\left(\frac{\log x}{2}\right) - \log\left(\frac{\cot x}{2}\right)$

(x) $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$ (xi) $\frac{e^{\sqrt{x}}+1}{e^{\sqrt{x}}-1}$

(xii) $\log[\tan^3 x \cdot \sin^4 x \cdot (x^2 + 7)^7]$

(xiii) $\log\left(\sqrt{\frac{1-\cos 3x}{1+\cos 3x}}\right)$

(xiv) $\log\left(\sqrt{\frac{1+\cos\left(\frac{5x}{2}\right)}{1-\cos\left(\frac{5x}{2}\right)}}\right)$

(xv) $\log\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right)$

(xvi) $\log\left[4^{2x}\left(\frac{x^2+5}{\sqrt{2x^3-4}}\right)^{\frac{3}{2}}\right]$

(xvii) $\log\left[\frac{e^{x^2}(5-4x)^{\frac{3}{2}}}{\sqrt[3]{7-6x}}\right]$

(xviii) $\log\left(\frac{a^{\cos x}}{(x^2-3)^3\log x}\right)$

(xix) $y = (25)^{\log_5(\sec x)} - (16)^{\log_4(\tan x)}$

(xx) $\frac{(x^2+2)^4}{\sqrt{x^2+5}}$

(4) A table of values of f, g, f' and g' is given

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	1	6	-3	4
4	3	4	5	-6
6	5	2	-4	7

(i) If $r(x) = f[g(x)]$ find $r'(2)$.

(ii) If $R(x) = g[3 + f(x)]$ find $R'(4)$.

(iii) If $s(x) = f[9 - f(x)]$ find $s'(4)$.

(iv) If $S(x) = g[g(x)]$ find $S'(6)$.

(5) Assume that $f'(3) = -1$, $g'(2) = 5$, $g(2) = 3$ and $y = f[g(x)]$ then $\left[\frac{dy}{dx}\right]_{x=2} = ?$

(6) If $h(x) = \sqrt{4f(x) + 3g(x)}$, $f(1) = 4$, $g(1) = 3$, $f'(1) = 3$, $g'(1) = 4$ find $h'(1)$.

(7) Find the x co-ordinates of all the points on the curve $y = \sin 2x - 2\sin x$, $0 \le x < 2\pi$ where $\frac{dy}{dx} = 0$.

(8) Select the appropriate hint from the hint basket and fill up the blank spaces in the following paragraph. [Activity]

"Let $f(x) = x^2 + 5$ and $g(x) = e^x + 3$ then

$f[g(x)]$ = _________ and

$g[f(x)]$ = _________.

Now $f'(x)$ = _________ and

$g'(x)$ = _________.

The derivative of $f[g(x)]$ w. r. t. x in terms of f and g is _________.

Therefore $\frac{d}{dx}[f[g(x)]]$ = __________ and

$\left[\frac{d}{dx}[f[g(x)]]\right]_{x=0}$ = ____________.

The derivative of $g[f(x)]$ w. r. t. x in terms of f and g is ____________.

Therefore $\frac{d}{dx}[g[f(x)]]$ = __________

and $\left[\frac{d}{dx}[g[f(x)]]\right]_{x=-1}$ = ____________."

Hint basket : $\{f'[g(x)]\cdot g'(x),\ 2e^{2x} + 6e^x,\ 8,\ g'[f(x)]\cdot f'(x),\ 2xe^{x^2+5},\ -2e^6,\ e^{2x} + 6e^x + 14,\ e^{x^2+5} + 3,\ 2x,\ e^x\}$

1.2.1 Geometrical meaning of Derivative :

Consider a point P on the curve $f(x)$. At $x = a$, the coordinates of P are $(a, f(a))$. Let Q be another point on the curve, a little to the right of P i.e. to the right of $x = a$, with a value increased by a small real number h. Therefore the coordinates of Q are $((a+h), f(a+h))$. Now we can calculate the slope of the secant line PQ i.e. slope of the secant line connecting the points P $(a, f(a))$ and Q $((a+h), f(a+h))$, by using formula for slope.

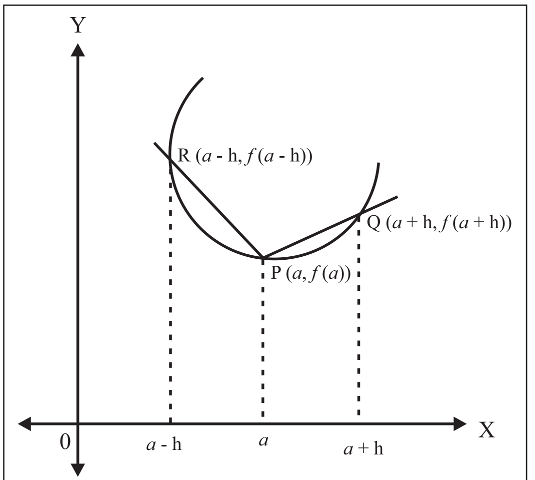

Fig. 1.2.1

Slope of secant PQ $= \dfrac{f(a+h) - f(a)}{a+h-a}$

$= \dfrac{f(a+h) - f(a)}{h}$

Suppose we make h smaller and smaller then $a + h$ will approach a as h gets closer to zero, Q will approach P, that is as $h \to 0$, the secant coverges to the tangent at P.

$$\therefore \quad \lim_{Q \to P} \text{(Slope of secant PQ)} = \lim_{h \to 0}\left[\frac{f(a+h) - f(a)}{h}\right] = f'(a)$$

So we get, Slope of tangent at P $= f'(a)$ ..... [If limit exists]

Thus the derivative of a function $y = f(x)$ at any point $P(x_1, y_1)$ is the slope of the tangent at that point on the curve. If we consider the point $a - h$ to the left of a, $h > 0$, then with R $= ((a-h), f(a-h))$ we will find the slope of PR which will also converge to the slope of tangent at P.

For Example : If $y = x^2 + 3x + 2$ then slope of the tangent at (2,3) is given by

Slope $m = \left[\dfrac{dy}{dx}\right]_{(2,3)} = \left[\dfrac{d}{dx}(x^2 + 3x + 2)\right]_{(2,3)} = (2x + 3)_{(2,3)} = 2(2) + 3$ $\quad\therefore\quad m = 7$

1.2.2 Derivatives of Inverse Functions :

We know that if $y = f(x)$ is a one-one and onto function then $x = f^{-1}(y)$ exists. If $f^{-1}(y)$ is differentiable then we can find its derivative. In this section let us discuss the derivatives of some inverse functions and the derivatives of inverse trigonometric functions.

Example 1 : Consider $f(x) = 2x - 2$ then its inverse is $f^{-1}(x) = \frac{x+2}{2}$. Let $g(x) = f^{-1}(x)$.

If we find the derivatives of these functions we see that $\frac{d}{dx}[f(x)] = 2$ and $\frac{d}{dx}[g(x)] = \frac{1}{2}$.

These derivatives are reciprocals of one another.

Example 2 : Consider $y = f(x) = x^2$. Let $g = f^{-1}$.

$\therefore\ g(y) = x = \sqrt{y}$

$\therefore\ g'(y) = \frac{1}{2\sqrt{y}}$ also $f'(x) = 2x$

Now $\frac{d}{dx}[g(f(x))] = \frac{f'(x)}{2\sqrt{f(x)}} = \frac{2x}{2\sqrt{x^2}} = 1$ and $g[f(x)] = x \therefore \frac{d}{dx}[g(f(x))] = \frac{d}{dx}(x) = 1$

At a point (x, x^2) on the curve, $f'(x) = 2x$ and $g'(y) = \frac{1}{2\sqrt{y}} = \frac{1}{2x} = \frac{1}{f'(x)}$.

1.2.3 Theorem : Suppose $y = f(x)$ is a differentiable function of x on an interval I and y is One-one, onto and $\frac{dy}{dx} \neq 0$ on I. Also if $f^{-1}(y)$ is differentiable on $f(I)$ then $\frac{d}{dy}[f^{-1}(y)] = \frac{1}{f'(x)}$ or $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ where $\frac{dy}{dx} \neq 0$.

Proof : Given that $y = f(x)$ and $x = f^{-1}(y)$ are differentiable functions.

Let there be a small increment in the value of x say δx then correspondingly there will be an increment in the value of y say δy. As δx and δy are increments, $\delta x \neq 0$ and $\delta y \neq 0$.

We have, $\frac{\delta x}{\delta y} \times \frac{\delta y}{\delta x} = 1$

$\therefore\ \frac{\delta x}{\delta y} = \frac{1}{\frac{\delta y}{\delta x}}$, where $\frac{\delta y}{\delta x} \neq 0$

Taking the limit as $\delta x \to 0$, we get,

$$\lim_{\delta x \to 0}\left(\frac{\delta x}{\delta y}\right) = \lim_{\delta x \to 0}\left(\frac{1}{\frac{\delta y}{\delta x}}\right)$$

as $\delta x \to 0$, $\delta y \to 0$,

$$\lim_{\delta y \to 0}\left(\frac{\delta x}{\delta y}\right) = \frac{1}{\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)} \qquad \ldots\ldots \text{(I)}$$

Since $y = f(x)$ is a differentiable function of x.

we have, $\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{dy}{dx}$ and $\frac{dy}{dx} \neq 0$ . . . . . (II)

From (I) and (II), we get

$$\lim_{\delta y \to 0}\left(\frac{\delta x}{\delta y}\right) = \frac{1}{\frac{dy}{dx}} \qquad \text{. . . . . (III)}$$

As $\frac{dy}{dx} \neq 0$, $\frac{1}{\frac{dy}{dx}}$ exists and is finite. $\therefore$ $\lim_{\delta y \to 0}\left(\frac{\delta x}{\delta y}\right) = \frac{dx}{dy}$ exists and is finite.

Hence, from (III) $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ where $\frac{dy}{dx} \neq 0$

An alternative proof using derivatives of composite functions rule.

We know that $f^{-1}[f(x)] = x$ [Identity function]

Taking derivative on bothe sides we get,

$$\frac{d}{dx}\left[f^{-1}[f(x)]\right] = \frac{d}{dx}(x)$$

i.e. $\left(f^{-1}\right)'[f(x)]\,\frac{d}{dx}[f(x)] = 1$

i.e. $\left(f^{-1}\right)'[f(x)]\,f'(x) = 1$

$\therefore$ $\left(f^{-1}\right)'[f(x)] = \frac{1}{f'(x)}$ . . . . . (I)

So, if $y = f(x)$ is a differentiable function of x and $x = f^{-1}(y)$ exists and is differentiable then

$$\left(f^{-1}\right)'[f(x)] = \left(f^{-1}\right)'(y) = \frac{dx}{dy} \text{ and } f'(x) = \frac{dy}{dx}$$

$\therefore$ (I) becomes

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \text{ where } \frac{dy}{dx} \neq 0$$

SOLVED EXAMPLES

Ex. 1 : Find the derivative of the function $y = f(x)$ using the derivative of the inverse function $x = f^{-1}(y)$ in the following

(i) $y = \sqrt[3]{x+4}$ (ii) $y = \sqrt{1+\sqrt{x}}$ (iii) $y = \ln x$

Solution :

(i) $y = \sqrt[3]{x+4}$

We first find the inverse of the function $y = f(x)$, i.e. x in term of y.

$y^3 = x + 4$ $\therefore$ $x = y^3 - 4$ $\therefore$ $x = f^{-1}(y) = y^3 - 4$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d}{dy}\left(y^3 - 4\right)} = \frac{1}{3y^2}$$

$$= \frac{1}{3\left(\sqrt[3]{x+4}\right)^2} = \frac{1}{3\sqrt[3]{(x+4)^2}} \qquad \text{for } x \neq -4$$

(ii) $y = \sqrt{1+\sqrt{x}}$

We first find the inverse of the function $y = f(x)$, i.e. x in term of y.

$y^2 = 1 + \sqrt{x}$ i.e. $\sqrt{x} = y^2 - 1$, $\therefore\ x = f^{-1}(y) = (y^2 - 1)^2$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d}{dy}\left[(y^2-1)^2\right]} = \frac{1}{2(y^2-1)\frac{d}{dy}(y^2-1)}$$

$$= \frac{1}{2(y^2-1)(2y)} = \frac{1}{4\sqrt{1+\sqrt{x}}\left[\left(\sqrt{1+\sqrt{x}}\right)^2 - 1\right]}$$

$$= \frac{1}{4\sqrt{1+\sqrt{x}}\,(1+\sqrt{x}-1)} = \frac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}$$

(iii) $y = \log x$

We first find the inverse of the function $y = f(x)$, i.e. x in term of y.

$y = \log x\ \therefore\ x = f^{-1}(y) = e^y$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d}{dy}(e^y)} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Ex. 2 : Find the derivative of the inverse of function $y = 2x^3 - 6x$ and calculate its value at $x = -2$.

Solution : Given : $y = 2x^3 - 6x$

Diff. *w. r. t.* x we get,

$$\frac{dy}{dx} = 6x^2 - 6 = 6(x^2 - 1)$$

we have, $\dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}$

$$\therefore\ \frac{dx}{dy} = \frac{1}{6(x^2-1)}$$

at $x = -2$,

we get, $y = 2(-2)^3 - 6(-2)$

$= -16 + 12 = -4$

$$\left[\frac{dx}{dy}\right]_{y=-4} = \frac{1}{\left[\frac{dy}{dx}\right]_{x=-2}}$$

$$= \frac{1}{6\left((-2)^2 - 1\right)}$$

$$= \frac{1}{18}$$

Ex. 3 : Let f and g be the inverse functions of each other. The following table lists a few values of f, g and f'

x	$f(x)$	$g(x)$	$f'(x)$
-4	2	1	$\frac{1}{3}$
1	-4	-2	4

find $g'(-4)$.

Solution : In order to find $g'(-4)$, we should first find an expression for $g'(x)$ for any input x. Since f and g are inverses we can use the following identify which holds for any two diffetentiable inverse functions.

$g'(x) = \dfrac{1}{f'[g(x)]}$ ... [check, how?]

... $[\text{Hint} : f[g(x)] = x]$

$$\therefore\ g'(-4) = \frac{1}{f'[g(-4)]}$$

$$= \frac{1}{f'(1)} = \frac{1}{4}$$

Ex. 4 : Let $f(x) = x^5 + 2x - 3$. Find $\left(f^{-1}\right)'(-3)$.

Solution : Given : $f(x) = x^5 + 2x - 3$

Diff. *w. r. t.* x we get,

$f'(x) = 5x^4 + 2$

Note that $f(x) = -3$ corresponds to $x = 0$.

$$\therefore \quad \left(f^{-1}\right)'(-3) = \frac{1}{f'(0)}$$

$$= \frac{1}{5(0)+2} = \frac{1}{2}$$

1.2.4 Derivatives of Standard Inverse trigononmetric Functions :

We observe that inverse trigonometric functions are multi-valued functions and because of this, their derivatives depend on which branch of the function we are dealing with. We are not restricted to use these branches all the time. While solving the problems it is customary to select the branch of the inverse trigonometric function which is applicable to the kind of problem we are solving. We have to pay more attention towards the domain and range.

1. If $y = \sin^{-1} x, -1 \le x \le 1, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$ then prove that $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$, $|x| < 1$.

Proof : Given that $y = \sin^{-1} x, -1 \le x \le 1, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$$\therefore \quad x = \sin y \qquad \ldots . \text{(I)}$$

Differentiate *w. r. t.* y

$$\frac{dx}{dy} = \frac{d}{dy}(\sin y)$$

$$\frac{dx}{dy} = \cos y = \pm\sqrt{\cos^2 y} = \pm\sqrt{1-\sin^2 y}$$

$$\therefore \quad \frac{dx}{dy} = \pm\sqrt{1-x^2} \qquad \ldots . [\because \sin y = x]$$

But $\cos y$ is positive since y lies in 1st or 4th quadrant as $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$$\therefore \quad \frac{dx}{dy} = \sqrt{1-x^2}$$

We have $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \ |x| < 1$$

2. If $y = \cos^{-1} x, -1 \le x \le 1, 0 \le y \le \pi$ then prove that $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$.

[As home work for students to prove.]

3. If $y = \cot^{-1} x, x \in R, 0 < y < \pi$ then $\dfrac{dy}{dx} = -\dfrac{1}{1+x^2}$.

Proof : Given that $y = \cot^{-1} x, x \in R, 0 < y < \pi$

$\therefore \quad x = \cot y \qquad \ldots . (I)$

Differentiate *w. r. t. y*

$\dfrac{dx}{dy} = \dfrac{d}{dy}(\cot y)$

$\dfrac{dx}{dy} = -\operatorname{cosec}^2 y = -(1 + \cot^2 y)$

$\therefore \quad \dfrac{dx}{dy} = -(1 + x^2) \qquad \ldots . [\because \cot y = x]$

We have $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{-(1+x^2)} \qquad \therefore \quad \dfrac{dy}{dx} = -\dfrac{1}{1+x^2}$

4. If $y = \tan^{-1} x, x \in R, -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$ then $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$. [left as home work for students to prove.]

5. If $y = \sec^{-1} x$, such that $|x| \geq 1$ and $0 \leq y \leq \pi, y \neq \dfrac{\pi}{2}$ then $\dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}}$ if $x > 1$

$\dfrac{dy}{dx} = -\dfrac{1}{x\sqrt{x^2-1}}$ if $x < -1$

Proof : Given that $y = \sec^{-1} x, |x| \geq 1$ and $0 \leq y \leq \pi, y \neq \dfrac{\pi}{2}$

$\therefore \quad x = \sec y \qquad \ldots . (I)$

Differentiate *w. r. t. y*

$\dfrac{dx}{dy} = \dfrac{d}{dy}(\sec y)$

$\dfrac{dx}{dy} = \sec y \cdot \tan y$

$\therefore \quad \dfrac{dx}{dy} = \pm \sec y \cdot \sqrt{\tan^2 y}$

$= \pm \sec y \cdot \sqrt{\sec^2 y - 1}$

$\therefore \quad \dfrac{dx}{dy} = \pm x \sqrt{x^2 - 1} \qquad \ldots .. [\because \sec y = x]$

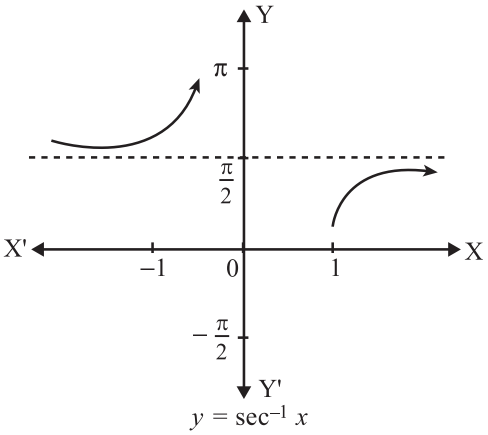

Fig. 1.2.2

We use the sign $\pm$ because for y in 1st and 2nd quadrant. $\sec y \cdot \tan y > 0$.

Hence we choose $x\sqrt{x^2-1}$ if $x > 1$ and $-x\sqrt{x^2-1}$ if $x < -1$

In 1st quadrant both $\sec y$ and $\tan y$ are positive.

In 2nd quadrant both $\sec y$ and $\tan y$ are negative.

$\therefore$ $\sec y \cdot \tan y$ is positive in both first and second quadrant.

Also, for $x > 0$, $x\sqrt{x^2-1} > 0$

and for $x < 0$, $-x\sqrt{x^2-1} > 0$

$\frac{dx}{dy} = x\sqrt{x^2-1}$, when $x > 0$, $|x| > 1$ i.e. $x > 1$

$= -x\sqrt{x^2-1}$, when $x < 0$, $|x| > 1$ i.e. $x < -1$

$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$ if $x > 1$

$\frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}}$ if $x < -1$

Note 1 : A function is increasing if its derivative is positive and is decreasing if its derivative is negative.

Note 2 : The derivative of $\sec^{-1} x$ is always positive because the graph of $\sec^{-1} x$ is always increasing.

6. If $y = -\operatorname{cosec} x$, such that $|x| \geq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$ then $\frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}}$ if $x > 1$

$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$ if $x < -1$

[Left as home work for students to prove]

Note 3 : The derivative of $\operatorname{cosec}^{-1} x$ is always negative because the graph of $\operatorname{cosec}^{-1} x$ is always decreasing.

1.2.5 Derivatives of Standard Inverse trigonometric Functions :

y	$\frac{dy}{dx}$	Conditions
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$, $\lvert x\rvert < 1$	$-1 \leq x \leq 1$ $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$, $\lvert x\rvert < 1$	$-1 \leq x \leq 1$ $0 \leq y \leq \pi$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$x \in R$ $-\frac{\pi}{2} < y < \frac{\pi}{2}$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$	$x \in R$ $0 < y < \pi$

y	$\frac{dy}{dx}$	Conditions
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$ for $x > 1$ $-\frac{1}{x\sqrt{x^2-1}}$ for $x < -1$	$\lvert x\rvert \geq 1$ $0 \leq y \leq \pi$ $y \neq \frac{\pi}{2}$
$\operatorname{cosec}^{-1} x$	$-\frac{1}{x\sqrt{x^2-1}}$ for $x > 1$ $\frac{1}{x\sqrt{x^2-1}}$ for $x < -1$	$\lvert x\rvert \geq 1$ $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ $y \neq 0$

Table 1.2.1

1.2.6 Derivatives of Standard Inverse trigonometric Composite Functions :

y	$\frac{dy}{dx}$
$\sin^{-1}[f(x)]$	$\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$, $\lvert f(x)\rvert < 1$
$\cos^{-1}[f(x)]$	$-\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$, $\lvert f(x)\rvert < 1$
$\tan^{-1}[f(x)]$	$\frac{f'(x)}{1+[f(x)]^2}$

y	$\frac{dy}{dx}$
$\cot^{-1}[f(x)]$	$-\frac{f'(x)}{1+[f(x)]^2}$
$\sec^{-1}[f(x)]$	$\frac{f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$, for $\lvert f(x)\rvert > 1$
$\mathrm{cosec}^{-1}[f(x)]$	$-\frac{f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$, for $\lvert f(x)\rvert > 1$

Table 1.2.2

Some Important Formulae for Inverse Trigonometric Functions :

(1) $\sin^{-1}(\sin\theta) = \theta$, $\sin(\sin^{-1}x) = x$	(2) $\cos^{-1}(\cos\theta) = \theta$, $\cos(\cos^{-1}x) = x$	
(3) $\tan^{-1}(\tan\theta) = \theta$, $\tan(\tan^{-1}x) = x$	(4) $\cot^{-1}(\cot\theta) = \theta$, $\cot(\cot^{-1}x) = x$	
(5) $\sec^{-1}(\sec\theta) = \theta$, $\sec(\sec^{-1}x) = x$	(6) $\mathrm{cosec}^{-1}(\mathrm{cosec}\,\theta) = \theta$, $\mathrm{cosec}(\mathrm{cosec}^{-1}x) = x$	
(7) $\sin^{-1}(\cos\theta) = \sin^{-1}\left[\sin\left(\frac{\pi}{2}-\theta\right)\right] = \frac{\pi}{2}-\theta$	(8) $\cos^{-1}(\sin\theta) = \cos^{-1}\left[\cos\left(\frac{\pi}{2}-\theta\right)\right] = \frac{\pi}{2}-\theta$	
(9) $\tan^{-1}(\cot\theta) = \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\theta\right)\right] = \frac{\pi}{2}-\theta$	(10) $\cot^{-1}(\tan\theta) = \cot^{-1}\left[\cot\left(\frac{\pi}{2}-\theta\right)\right] = \frac{\pi}{2}-\theta$	
(11) $\sec^{-1}(\mathrm{cosec}\,\theta) = \sec^{-1}\left[\sec\left(\frac{\pi}{2}-\theta\right)\right] = \frac{\pi}{2}-\theta$		
(12) $\mathrm{cosec}^{-1}(\sec\theta) = \mathrm{cosec}^{-1}\left[\mathrm{cosec}\left(\frac{\pi}{2}-\theta\right)\right] = \frac{\pi}{2}-\theta$		
(13) $\sin^{-1}(x) = \mathrm{cosec}^{-1}\left(\frac{1}{x}\right)$	(14) $\mathrm{cosec}^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$	
(15) $\cos^{-1}(x) = \sec^{-1}\left(\frac{1}{x}\right)$	(16) $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$	
(17) $\tan^{-1}(x) = \cot^{-1}\left(\frac{1}{x}\right)$	(18) $\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right)$	
(19) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$	(20) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$	(21) $\sec^{-1}x + \mathrm{cosec}^{-1}x = \frac{\pi}{2}$
(22) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$	(23) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$	

In above tables, x is a real variable with restrictions. **Table 1.2.3**

Some Important Substitutions :

Expression	Substitutions
$\sqrt{1-x^2}$	$x = \sin\theta$ or $x = \cos\theta$
$\sqrt{1-x^2}$	$x = \tan\theta$ or $x = \cot\theta$
$\sqrt{x^2+1}$	$x = \sec\theta$ or $x = \mathrm{cosec}\,\theta$
$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a\cos 2\theta$ or $x = a\cos\theta$
$\sqrt{\frac{1+x}{1-x}}$ or $\sqrt{\frac{1-x}{1+x}}$	$x = \cos 2\theta$ or $x = \cos\theta$
$\sqrt{\frac{a+x^2}{a-x^2}}$ or $\sqrt{\frac{a-x^2}{a+x^2}}$	$x^2 = a\cos 2\theta$ or $x^2 = a\cos\theta$

Expression	Substitutions
$\frac{2x}{1+x^2}$	$x = \tan\theta$
$\frac{1-x^2}{1+x^2}$	$x = \tan\theta$
$3x-4x^3$ or $1-2x^2$	$x = \sin\theta$
$4x^3-3x$ or $2x^2-1$	$x = \cos\theta$
$\frac{3x-x^3}{1-3x^2}$	$x = \tan\theta$
$\frac{2f(x)}{1+[f(x)]^2}$ or $\frac{1-[f(x)]^2}{1+[f(x)]^2}$	$f(x) = \tan\theta$

Table 1.2.4

SOLVED EXAMPLES

Ex. 1 : Using derivative prove that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$.

Solution : Let $f(x) = \sin^{-1}x + \cos^{-1}x$ . . . . . (I)

We have to prove that $f(x) = \frac{\pi}{2}$

Differentiate (I) *w. r. t. x*

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}[\sin^{-1}x + \cos^{-1}x]$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

$f'(x) = 0 \Rightarrow f(x)$ is a constant function.

Let $f(x) = c$. For any value of x, $f(x)$ must be c only. So conveniently we can choose $x = 0$,

$\therefore$ from (I) we get,

$$f(0) = \sin^{-1}(0) + \cos^{-1}(0) = 0 + \frac{\pi}{2} = \frac{\pi}{2} \Rightarrow c = \frac{\pi}{2} \;\therefore\; f(x) = \frac{\pi}{2}$$

Hence, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$.

Ex. 2 : Differentiate the following *w. r. t. x*.

(i) $\sin^{-1}(x^3)$ (ii) $\cos^{-1}(2x^2 - x)$ (iii) $\sin^{-1}(2^x)$

(iv) $\cot^{-1}\left(\frac{1}{x^2}\right)$ (v) $\cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$ (vi) $\sin^2\left(\sin^{-1}(x^2)\right)$

Solution :

(i) Let $y = \sin^{-1}(x^3)$

Differentiate w. r. t. x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1}(x^3)\right)$$

$$= \frac{1}{\sqrt{1-(x^3)^2}} \cdot \frac{d}{dx}(x^3)$$

$$= \frac{1}{\sqrt{1-x^6}}(3x^2)$$

$$\therefore \frac{dy}{dx} = \frac{3x^2}{\sqrt{1-x^6}}$$

(iii) Let $y = \sin^{-1}(2^x)$

Differentiate w. r. t. x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1}(2^x)\right)$$

$$= \frac{1}{\sqrt{1-(2^x)^2}} \cdot \frac{d}{dx}(2^x)$$

$$= \frac{1}{\sqrt{1-2^{2x}}}(2^x \log 2)$$

$$\therefore \frac{dy}{dx} = \frac{2^x \log 2}{\sqrt{1-4^x}}$$

(iv) Let $y = \cot^{-1}\left(\frac{1}{x^2}\right) = \tan^{-1}(x^2)$

Differentiate w. r. t. x.

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}(x^2))$$

$$= \frac{1}{1+(x^2)^2} \cdot \frac{d}{dx}(x^2)$$

$$\therefore \frac{dy}{dx} = \frac{2x}{1+x^4}$$

(vi) Let $y = \sin^2\left(\sin^{-1}(x^2)\right)$

$$= [\sin(\sin^{-1}(x^2))]^2 \quad = (x^2)^2$$

$$\therefore \quad y = x^4$$

Differentiate w. r. t. x.

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) \qquad \therefore \frac{dy}{dx} = 4x^3$$

(ii) Let $y = \cos^{-1}(2x^2 - x)$

Hence $\cos y = 2x^2 - x$ . . . (I)

Differentiate w. r. t. x.

$$-\sin y \cdot \frac{dy}{dx} = 4x - 1$$

$$\frac{dy}{dx} = \frac{1-4x}{\sin y} = \frac{1-4x}{\sqrt{1-\cos^2 y}}$$

$$\therefore \frac{dy}{dx} = \frac{1-4x}{\sqrt{1-x^2(2x-1)^2}} \quad \text{. . . from (I)}$$

Alternate Method :

If $y = \cos^{-1}(2x^2 - x)$

Differentiate w. r. t. x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos^{-1}(2x^2 - x)\right)$$

$$= \frac{-1}{\sqrt{1-(2x^2-x)^2}} \cdot \frac{d}{dx}(2x^2 - x)$$

$$= \frac{-1}{\sqrt{1-x^2(2x-1)^2}} \cdot (4x-1)$$

$$\therefore \frac{dy}{dx} = \frac{1-4x}{\sqrt{1-x^2(2x-1)^2}}$$

(v) Let $y = \cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$

Differentiate w. r. t. x.

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)\right]$$

$$= -\frac{1}{\sqrt{1-\left(\sqrt{\frac{1+x}{2}}\right)^2}} \cdot \frac{d}{dx}\left(\sqrt{\frac{1+x}{2}}\right)$$

$$= -\frac{1}{\sqrt{1-\frac{1+x}{2}}} \times \frac{1}{2\sqrt{\frac{1+x}{2}}} \times \frac{d}{dx}\left(\frac{1+x}{2}\right)$$

$$= -\frac{\sqrt{2}}{\sqrt{1-x}} \times \frac{1}{\sqrt{2}\sqrt{1+x}} \times \frac{1}{2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2\sqrt{1-x^2}}$$

Ex. 3 : Differentiate the following *w. r. t. x*.

(i) $\cos^{-1}(4\cos^3 x - 3\cos x)$ (ii) $\cos^{-1}[\sin(4^x)]$ (iii) $\sin^{-1}\left(\sqrt{\dfrac{1-\cos x}{2}}\right)$

(iv) $\tan^{-1}\left(\dfrac{1-\cos 3x}{\sin 3x}\right)$ (v) $\cot^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$

Solution :

(i) Let $y = \cos^{-1}(4\cos^3 x - 3\cos x)$

$= \cos^{-1}(\cos 3x)$

$\therefore \quad y = 3x$

Differentiate *w. r. t. x*.

$\dfrac{dy}{dx} = \dfrac{d}{dx}(3x)$

$\therefore \dfrac{dy}{dx} = 3$

(ii) Let $y = \cos^{-1}[\sin(4^x)]$

$= \cos^{-1}\left[\cos\left(\dfrac{\pi}{2} - 4^x\right)\right]$

$\therefore \quad y = \dfrac{\pi}{2} - 4^x$

Differentiate *w. r. t. x*.

$\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\pi}{2} - 4^x\right) = 0 - 4^x \log 4$

$\therefore \dfrac{dy}{dx} = -4^x \log 4$

(iii) Let $y = \sin^{-1}\left(\sqrt{\dfrac{1-\cos x}{2}}\right)$

$= \sin^{-1}\left(\sqrt{\dfrac{2\sin^2\left(\frac{x}{2}\right)}{2}}\right)$

$= \sin^{-1}\left[\sin\left(\dfrac{x}{2}\right)\right]$

$\therefore \quad y = \dfrac{x}{2}$

Differentiate *w. r. t. x*.

$\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x}{2}\right) \qquad \therefore \dfrac{dy}{dx} = \dfrac{1}{2}$

(iv) Let $y = \tan^{-1}\left(\dfrac{1-\cos 3x}{\sin 3x}\right)$

$= \tan^{-1}\left(\dfrac{2\sin^2\left(\frac{3x}{2}\right)}{2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)}\right)$

$= \tan^{-1}\left[\tan\left(\dfrac{3x}{2}\right)\right]$

$\therefore \quad y = \dfrac{3x}{2}$

Differentiate *w. r. t. x*.

$\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{3x}{2}\right) \qquad \therefore \dfrac{dy}{dx} = \dfrac{3}{2}$

(v) Let $y = \cot^{-1}\left(\dfrac{\cos x}{1+\sin x}\right) = \tan^{-1}\left(\dfrac{1+\sin x}{\cos x}\right)$

$= \tan^{-1}\left(\dfrac{\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]^2}{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}\right) = \tan^{-1}\left(\dfrac{\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]^2}{\left[\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right]\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]}\right)$

$= \tan^{-1}\left(\dfrac{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}\right) = \tan^{-1}\left(\dfrac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right)$ ....Divide Numerator & Denominator by $\cos\left(\dfrac{x}{2}\right)$

$= \tan^{-1}\left[\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\right] \qquad \therefore y = \dfrac{\pi}{4}+\dfrac{x}{2}$

Differentiate *w. r. t. x*.

$\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right) = 0 + \dfrac{1}{2} \qquad \therefore \dfrac{dy}{dx} = \dfrac{1}{2}$

Ex. 4 : Differentiate the following *w. r. t. x*.

(i) $\sin^{-1}\left(\dfrac{2\cos x + 3\sin x}{\sqrt{13}}\right)$ (ii) $\cos^{-1}\left(\dfrac{3\sin x^2 + 4\cos x^2}{5}\right)$ (iii) $\sin^{-1}\left(\dfrac{a\cos x - b\sin x}{\sqrt{a^2+b^2}}\right)$

Solution :

(i) Let $y = \sin^{-1}\left(\dfrac{2\cos x + 3\sin x}{\sqrt{13}}\right)$

$$= \sin^{-1}\left(\frac{2}{\sqrt{13}}\cos x + \frac{3}{\sqrt{13}}\cos x\right)$$

Put $\dfrac{2}{\sqrt{13}} = \sin\alpha,\ \dfrac{3}{\sqrt{13}} = \cos\alpha$

Also, $\sin^2\alpha + \cos^2\alpha = \dfrac{4}{3} + \dfrac{9}{13} = 1$

And $\tan\alpha = \dfrac{2}{3}\ \therefore\ \alpha = \tan^{-1}\left(\dfrac{2}{3}\right)$

$$y = \sin^{-1}(\sin\alpha\cos x + \cos\alpha\sin x)$$

$$y = \sin^{-1}(\sin x\cos\alpha + \cos x\sin\alpha)$$

$$y = \sin^{-1}[\sin(x+\alpha)]$$

$$\therefore\quad y = x + \tan^{-1}\left(\frac{2}{3}\right)$$

Differentiate *w. r. t. x*.

$$\frac{dy}{dx} = \frac{d}{dx}\left[x + \tan^{-1}\left(\frac{2}{3}\right)\right] = 1 + 0$$

$$\therefore\quad \frac{dy}{dx} = 1$$

(ii) Let $y = \cos^{-1}\left(\dfrac{3\sin x^2 + 4\cos x^2}{5}\right)$

$$= \cos^{-1}\left(\frac{3}{5}\sin x^2 + \frac{4}{5}\cos x^2\right)$$

Put $\dfrac{3}{5} = \sin\alpha,\ \dfrac{4}{5} = \cos\alpha$

Also, $\sin^2\alpha + \cos^2\alpha = \dfrac{9}{25} + \dfrac{16}{25} = 1$

And $\tan\alpha = \dfrac{3}{4}\ \therefore\ \alpha = \tan^{-1}\left(\dfrac{3}{4}\right)$

$$y = \cos^{-1}(\sin\alpha\sin x^2 + \cos\alpha\cos x^2)$$

$$y = \cos^{-1}(\cos x^2\cos\alpha + \sin x^2\sin\alpha)$$

$$y = \cos^{-1}[\cos(x^2 - \alpha)]$$

$$\therefore\quad y = x^2 - \tan^{-1}\left(\frac{3}{4}\right)$$

Differentiate *w. r. t. x*.

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^2 - \tan^{-1}\left(\frac{3}{4}\right)\right] = 2x - 0$$

$$\therefore\quad \frac{dy}{dx} = 2x$$

(iii) Let $y = \sin^{-1}\left(\dfrac{a\cos x - b\sin x}{\sqrt{a^2+b^2}}\right) = \sin^{-1}\left(\dfrac{a}{\sqrt{a^2+b^2}}\cos x - \dfrac{b}{\sqrt{a^2+b^2}}\sin x\right)$

Put $\dfrac{a}{\sqrt{a^2+b^2}} = \sin\alpha,\ \dfrac{b}{\sqrt{a^2+b^2}} = \cos\alpha$

Also, $\sin^2\alpha + \cos^2\alpha = \dfrac{a^2}{a^2+b^2} + \dfrac{b^2}{a^2+b^2} = 1$ And $\tan\alpha = \dfrac{a}{b}\ \therefore\ \alpha = \tan^{-1}\left(\dfrac{a}{b}\right)$

$$y = \sin^{-1}(\sin\alpha\cos x - \cos\alpha\sin x)$$

But $\sin(\alpha - x) = \sin\alpha\cos x - \cos\alpha\sin x$

$$y = \sin^{-1}[\sin(\alpha - x)]$$

$$\therefore\quad y = \tan^{-1}\left(\frac{a}{b}\right) - x$$

Differentiate *w. r. t. x*.

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}\left(\frac{a}{b}\right) - x\right] = 0 - 1 \qquad \therefore\ \frac{dy}{dx} = -1$$

Ex. 5 : Differentiate the following *w. r. t. x*.

(i) $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ (ii) $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$ (iii) $\text{cosec}^{-1}\left(\dfrac{1}{3x-4x^3}\right)$

(iv) $\tan^{-1}\left(\dfrac{2e^x}{1-e^{2x}}\right)$ (v) $\cos^{-1}\left(\dfrac{1-9x^2}{1+9x^2}\right)$ (vi) $\cos^{-1}\left(\dfrac{2^x-2^{-x}}{2^x+2^{-x}}\right)$

(vii) $\tan^{-1}\left(\sqrt{\dfrac{3-x}{3+x}}\right)$ (viii) $\sin^{-1}\left(\dfrac{5\sqrt{1-x^2}-12x}{13}\right)$ (ix) $\sin^{-1}\left(\dfrac{2^{x+1}}{1+4^x}\right)$

Solution :

(i) Let $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Put $x = \tan\theta \therefore \theta = \tan^{-1}x$

$\therefore \quad y = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right)$

$y = \sin^{-1}(\sin 2\theta) = 2\theta$

$\therefore \quad y = 2\tan^{-1}x$

Differentiate *w. r. t. x*.

$\dfrac{dy}{dx} = 2\dfrac{d}{dx}(\tan^{-1}x)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{2}{1+x^2}$

(ii) Let $y = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$

Put $x = \sin\theta \therefore \theta = \sin^{-1}x$

$\therefore \quad y = \cos^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$y = \cos^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$

$y = \cos^{-1}(2\sin\theta\cos\theta) = \cos^{-1}(\sin 2\theta)$

$y = \cos^{-1}\left[\cos\left(\dfrac{\pi}{2}-2\theta\right)\right] = \dfrac{\pi}{2}-2\theta$

$\therefore \quad y = \dfrac{\pi}{2} - 2\sin^{-1}x$

Differentiate *w. r. t. x*.

$\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\pi}{2} - 2\sin^{-1}x\right)$

$\dfrac{dy}{dx} = 0 - \dfrac{2\times 1}{\sqrt{1-x^2}}$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{2}{\sqrt{1-x^2}}$

(iii) Let $y = \text{cosec}^{-1}\left(\dfrac{1}{3x-4x^3}\right)$

$y = \sin^{-1}(3x-4x^3)$

Put $x = \sin\theta \therefore \theta = \sin^{-1}x$

$y = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$y = \sin^{-1}(\sin 3\theta) = 3\theta$

$\therefore \quad y = 3\sin^{-1}x$

Differentiate *w. r. t. x*.

$\dfrac{dy}{dx} = 3\dfrac{d}{dx}(\sin^{-1}x)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{3}{\sqrt{1-x^2}}$

(iv) Let $y = \tan^{-1}\left(\dfrac{2e^x}{1-e^{2x}}\right)$

Put $e^x = \tan\theta \therefore \theta = \tan^{-1}(e^x)$

$y = \tan^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right)$

$y = \tan^{-1}(\tan 2\theta) = 2\theta$

$\therefore \quad y = 2\tan^{-1}(e^x)$

Differentiate *w. r. t. x*.

$\dfrac{dy}{dx} = 2\dfrac{d}{dx}[\tan^{-1}(e^x)]$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{2}{1+(e^x)^2}\dfrac{d}{dx}(e^x) = \dfrac{2e^x}{1-e^{2x}}$

(v) Let $y = \cos^{-1}\left(\dfrac{1-9x^2}{1+9x^2}\right)$

$y = \cos^{-1}\left(\dfrac{1-(3x)^2}{1+(3x)^2}\right)$

Put $3x = \tan\theta \therefore \theta = \tan^{-1}(3x)$

$\therefore \quad y = \cos^{-1}\left[\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right]$

$y = \cos^{-1}(\cos 2\theta) = 2\theta$

$y = 2\tan^{-1}(3x)$

Differentiate *w. r. t. x.*

$\dfrac{dy}{dx} = 2\dfrac{d}{dx}\left[\tan^{-1}(3x)\right]$

$\dfrac{dy}{dx} = \dfrac{2}{1+(3x)^2}\dfrac{d}{dx}(3x)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{6}{1+9x^2}$

(vi) Let $y = \cos^{-1}\left(\dfrac{2^x-2^{-x}}{2^x+2^{-x}}\right)$

$y = \cos^{-1}\left(\dfrac{2^x(2^x-2^{-x})}{2^x(2^x+2^{-x})}\right)$ . . . [Multiply & Devide by 2^x]

$y = \cos^{-1}\left(\dfrac{2^{2x}-1}{2^{2x}+1}\right) = \cos^{-1}\left[-\dfrac{1-(2^x)^2}{1+(2^x)^2}\right]$

Put $2^x = \tan\theta \therefore \theta = \tan^{-1}(2^x)$

$\therefore \quad y = \cos^{-1}\left[-\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right] = \cos^{-1}[-\cos 2\theta]$

$y = \cos^{-1}[\cos(\pi - 2\theta)] = \pi - 2\theta$

$y = \pi - 2\tan^{-1}(2^x)$

Differentiate *w. r. t. x.*

$\dfrac{dy}{dx} = \dfrac{d}{dx}\left[\pi - 2\tan^{-1}(2^x)\right]$

$\dfrac{dy}{dx} = 0 - \dfrac{2}{1+(2^x)^2}\dfrac{d}{dx}(2^x) = -\dfrac{2\cdot 2^x\cdot\log 2}{1+2^{2x}}$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{2^{x+1}\log 2}{1+2^{2x}}$

(vii) Let $y = \tan^{-1}\left(\sqrt{\dfrac{3-x}{3+x}}\right)$

Put $x = 3\cos 2\theta \therefore \theta = \dfrac{1}{2}\cos^{-1}\left(\dfrac{x}{3}\right)$

$\therefore \quad y = \tan^{-1}\left[\sqrt{\dfrac{3-3\cos 2\theta}{3+3\cos 2\theta}}\right] = \tan^{-1}\left[\sqrt{\dfrac{3(1-\cos 2\theta)}{3(1+\cos 2\theta)}}\right] = \tan^{-1}\left[\sqrt{\dfrac{2\sin^2\theta}{2\cos^2\theta}}\right]$

$y = \tan^{-1}\left(\sqrt{\tan^2\theta}\right) = \tan^{-1}(\tan\theta)$

$y = \theta = \dfrac{1}{2}\cos^{-1}\left(\dfrac{x}{3}\right)$

Differentiate *w. r. t. x.*

$\dfrac{dy}{dx} = \dfrac{1}{2}\cdot\dfrac{d}{dx}\left[\cos^{-1}\left(\dfrac{x}{3}\right)\right]$

$= \dfrac{1}{2}\left[-\dfrac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}}\right]\dfrac{d}{dx}\left(\dfrac{x}{3}\right) = -\dfrac{1}{2}\times\dfrac{1}{\sqrt{\frac{9-x^2}{9}}}\times\dfrac{1}{3}$

$= -\dfrac{1}{2}\times\dfrac{1}{\frac{\sqrt{9-x^2}}{3}}\times\dfrac{1}{3}$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{1}{2\sqrt{9-x^2}}$

(viii) Let $y = \sin^{-1}\left(\frac{5\sqrt{1-x^2} - 12x}{13}\right)$

Put $x = \sin\theta \therefore \theta = \sin^{-1}x$

$$\therefore \quad y = \sin^{-1}\left(\frac{5\sqrt{1-\sin^2\theta} - 12\sin\theta}{13}\right) = \sin^{-1}\left(\frac{5\sqrt{\cos^2\theta} - 12\sin\theta}{13}\right) = \sin^{-1}\left(\frac{5\cos\theta - 12\sin\theta}{13}\right)$$

$$\therefore \quad y = \sin^{-1}\left(\frac{5}{13}\cos\theta - \frac{12}{13}\sin\theta\right)$$

Put $\frac{5}{13} = \sin\alpha, \frac{12}{13} = \cos\alpha$

Also, $\sin^2\alpha + \cos^2\alpha = \frac{25}{169} + \frac{144}{169} = 1$

And $\tan\alpha = \frac{5}{12} \therefore \alpha = \tan^{-1}\left(\frac{5}{12}\right)$

$$y = \sin^{-1}(\sin\alpha\cos\theta - \cos\alpha\sin\theta) = \sin^{-1}[\sin(\alpha - \theta)] = (\alpha - \theta)$$

$$\therefore \quad y = \tan^{-1}\left(\frac{5}{12}\right) - \sin^{-1}x$$

Differentiate *w. r. t. x.*

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}\left(\frac{5}{12}\right) - \sin^{-1}x\right] = 0 - \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

(ix) Let $y = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right) = \sin^{-1}\left(\frac{2 \cdot 2^x}{1+(2^x)^2}\right)$

Put $2^x = \tan\theta \therefore \theta = \tan^{-1}(2^x)$

$$\therefore \quad y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}(2^x)$$

Differentiate *w. r. t. x.*

$$\frac{dy}{dx} = 2\frac{d}{dx}[\tan^{-1}(2^x)] = \frac{2}{1+(2^x)^2} \cdot \frac{d}{dx}(2^x) = \frac{2}{1+2^{2x}}(2^x \cdot \log 2)$$

$$\therefore \quad \frac{dy}{dx} = -\frac{2^{x+1}\log 2}{1+4^x}$$

Ex. 6 : Differentiate the following *w. r. t. x.*

(i) $\tan^{-1}\left(\frac{4x}{1+21x^2}\right)$ (ii) $\tan^{-1}\left(\frac{7x}{1-12x^2}\right)$

(iii) $\cot^{-1}\left(\frac{b\sin x - a\cos x}{a\sin x + b\cos x}\right)$ (iv) $\tan^{-1}\left(\frac{5x+1}{3-x-6x^2}\right)$

Solution :

(i) Let $y = \tan^{-1}\left(\dfrac{4x}{1+21x^2}\right)$

$= \tan^{-1}\left(\dfrac{7x-3x}{1+(7x)(3x)}\right)$

$y = \tan^{-1}(7x) - \tan^{-1}(3x)$

Differentiate $w.\ r.\ t.\ x.$

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(7x) - \tan^{-1}(3x)]$$

$$= \frac{d}{dx}[\tan^{-1}(7x)] - \frac{d}{dx}[\tan^{-1}(3x)]$$

$$= \frac{1}{1+(7x)^2}\cdot\frac{d}{dx}(7x) - \frac{1}{1+(3x)^2}\cdot\frac{d}{dx}(3x)$$

$$\therefore \frac{dy}{dx} = \frac{7}{1+49x^2} - \frac{3}{1+9x^2}$$

(ii) Let $y = \tan^{-1}\left(\dfrac{7x}{1-12x^2}\right)$

$= \tan^{-1}\left(\dfrac{3x+4x}{1-(3x)(4x)}\right)$

$y = \tan^{-1}(3x) + \tan^{-1}(4x)$

Differentiate $w.\ r.\ t.\ x.$

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(3x) + \tan^{-1}(4x)]$$

$$= \frac{d}{dx}[\tan^{-1}(3x)] + \frac{d}{dx}[\tan^{-1}(4x)]$$

$$= \frac{1}{1+(3x)^2}\cdot\frac{d}{dx}(3x) + \frac{1}{1+(4x)^2}\cdot\frac{d}{dx}(4x)$$

$$\therefore \frac{dy}{dx} = \frac{3}{1+9x^2} + \frac{4}{1+16x^2}$$

(iii) Let $y = \cot^{-1}\left(\dfrac{b\sin x - a\cos x}{a\sin x + b\cos x}\right) = \tan^{-1}\left(\dfrac{a\sin x + b\cos x}{b\sin x - a\cos x}\right) = \tan^{-1}\left(\dfrac{\frac{a}{b} + \cot x}{1 - \left(\frac{a}{b}\right)(\cot x)}\right)$

$= \tan^{-1}\left(\dfrac{a}{b}\right) + \tan^{-1}(\cot x) = \tan^{-1}\left(\dfrac{a}{b}\right) + \tan^{-1}\left[\tan\left(\dfrac{\pi}{2} - x\right)\right]$

$y = \tan^{-1}\left(\dfrac{a}{b}\right) + \dfrac{\pi}{2} - x$

Differentiate $w.\ r.\ t.\ x.$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}\left(\frac{a}{b}\right) + \frac{\pi}{2} - x\right]$$

$$= \frac{d}{dx}\left[\tan^{-1}\left(\frac{a}{b}\right)\right] + \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{d}{dx}(x)$$

$$= 0 + 0 - 1$$

$$\therefore \frac{dy}{dx} = -1$$

(iv) Let $y = \tan^{-1}\left(\dfrac{5x+1}{3-x-6x^2}\right) = \tan^{-1}\left(\dfrac{5x+1}{1+2-x-6x^2}\right) = \tan^{-1}\left(\dfrac{5x+1}{1-(6x^2+x-2)}\right)$

$= \tan^{-1}\left(\dfrac{5x+1}{1-(6x^2+4x-3x-2)}\right) = \tan^{-1}\left(\dfrac{5x+1}{1-[2x(3x+2)-(3x+2)]}\right)$

$= \tan^{-1}\left(\dfrac{5x+1}{1-(3x+2)(2x-1)}\right) = \tan^{-1}\left(\dfrac{(3x+2)+(2x-1)}{1-(3x+2)(2x-1)}\right)$

$y = \tan^{-1}(3x+2) + \tan^{-1}(2x-1)$

Differentiate *w. r. t. x.*

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(3x+2) + \tan^{-1}(2x-1)]$$

$$= \frac{d}{dx}[\tan^{-1}(3x+2)] + \frac{d}{dx}[\tan^{-1}(2x-1)]$$

$$= \frac{1}{1+(3x+2)^2}\cdot\frac{d}{dx}(3x+2) + \frac{1}{1+(2x-1)^2}\cdot\frac{d}{dx}(2x-1)$$

$$\therefore \quad \frac{dy}{dx} = \frac{3}{1+(3x+2)^2} + \frac{2}{1+(2x-1)^2}$$

EXERCISE 1.2

(1) Find the derivative of the function $y = f(x)$ using the derivative of the inverse function $x = f^{-1}(y)$ in the following

(i) $y = \sqrt{x}$ (ii) $y = \sqrt{2-\sqrt{x}}$

(iii) $y = \sqrt[3]{x-2}$ (iv) $y = \log(2x-1)$

(v) $y = 2x+3$ (vi) $y = e^x - 3$

(vii) $y = e^{2x-3}$ (viii) $y = \log_2\left(\frac{x}{2}\right)$

(2) Find the derivative of the inverse function of the following

(i) $y = x^2\cdot e^x$ (ii) $y = x\cos x$

(iii) $y = x\cdot 7^x$ (iv) $y = x^2 + \log x$

(v) $y = x\log x$

(3) Find the derivative of the inverse of the following functions, and also find their value at the points indicated against them.

(i) $y = x^5 + 2x^3 + 3x$, at $x = 1$

(ii) $y = e^x + 3x + 2$, at $x = 0$

(iii) $y = 3x^2 + 2\log x^3$, at $x = 1$

(iv) $y = \sin(x-2) + x^2$, at $x = 2$

(4) If $f(x) = x^3 + x - 2$, find $(f^{-1})'(0)$.

(5) Using derivative prove

(i) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

(ii) $\sec^{-1}x + \mathrm{cosec}^{-1}x = \frac{\pi}{2}$ . . . [for $|x| \geq 1$]

(6) Differentiate the following *w. r. t. x.*

(i) $\tan^{-1}(\log x)$ (ii) $\mathrm{cosec}^{-1}(e^{-x})$

(iii) $\cot^{-1}(x^3)$ (iv) $\cot^{-1}(4^x)$

(v) $\tan^{-1}(\sqrt{x})$ (vi) $\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)$

(vii) $\cos^{-1}(1-x^2)$ (viii) $\sin^{-1}\left(x^{\frac{3}{2}}\right)$

(ix) $\cos^3[\cos^{-1}(x^3)]$ (x) $\sin^4[\sin^{-1}(\sqrt{x})]$

(7) Differentiate the following *w. r. t. x.*

(i) $\cot^{-1}[\cot(e^{x^2})]$

(ii) $\mathrm{cosec}^{-1}\left(\frac{1}{\cos(5^x)}\right)$

(iii) $\cos^{-1}\left(\sqrt{\frac{1+\cos x}{2}}\right)$

(iv) $\cos^{-1}\left(\sqrt{\frac{1-\cos(x^2)}{2}}\right)$

(v) $\tan^{-1}\left(\frac{1-\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\right)$

(vi) $\mathrm{cosec}^{-1}\left(\frac{1}{4\cos^3 2x - 3\cos 2x}\right)$

(vii) $\tan^{-1}\left(\frac{1+\cos\left(\frac{x}{3}\right)}{\sin\left(\frac{x}{3}\right)}\right)$

(viii) $\cot^{-1}\left(\frac{\sin 3x}{1+\cos 3x}\right)$

(ix) $\tan^{-1}\left(\dfrac{\cos 7x}{1+\sin 7x}\right)$

(x) $\tan^{-1}\left(\sqrt{\dfrac{1+\cos x}{1-\cos x}}\right)$

(xi) $\tan^{-1}(\operatorname{cosec} x + \cot x)$

(xii) $\cot^{-1}\left(\dfrac{\sqrt{1+\sin\left(\frac{4x}{3}\right)}+\sqrt{1-\sin\left(\frac{4x}{3}\right)}}{\sqrt{1+\sin\left(\frac{4x}{3}\right)}-\sqrt{1-\sin\left(\frac{4x}{3}\right)}}\right)$

(8) Differentiate the following *w. r. t. x*.

(i) $\sin^{-1}\left(\dfrac{4\sin x+5\cos x}{\sqrt{41}}\right)$

(ii) $\cos^{-1}\left(\dfrac{\sqrt{3}\cos x-\sin x}{2}\right)$

(iii) $\sin^{-1}\left(\dfrac{\cos\sqrt{x}+\sin\sqrt{x}}{\sqrt{2}}\right)$

(iv) $\cos^{-1}\left(\dfrac{3\cos 3x-4\sin 3x}{5}\right)$

(v) $\cos^{-1}\left(\dfrac{3\cos(e^x)+2\sin(e^x)}{\sqrt{13}}\right)$

(vi) $\operatorname{cosec}^{-1}\left(\dfrac{10}{6\sin(2^x)-8\cos(2^x)}\right)$

(9) Differentiate the following *w. r. t. x*.

(i) $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ (ii) $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

(iii) $\sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ (iv) $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

(v) $\cos^{-1}(3x-4x^3)$ (vi) $\cos^{-1}\left(\dfrac{e^x-e^{-x}}{e^x+e^{-x}}\right)$

(vii) $\cos^{-1}\left(\dfrac{1-9^x}{1+9^x}\right)$ (viii) $\sin^{-1}\left(\dfrac{4^{x+\frac{1}{2}}}{1+2^{4x}}\right)$

(ix) $\sin^{-1}\left(\dfrac{1-25x^2}{1+25x^2}\right)$ (x) $\sin^{-1}\left(\dfrac{1-x^3}{1+x^3}\right)$

(xi) $\tan^{-1}\left(\dfrac{2x^{\frac{5}{2}}}{1-x^5}\right)$ (xii) $\cot^{-1}\left(\dfrac{1-\sqrt{x}}{1+\sqrt{x}}\right)$

(10) Differentiate the following *w. r. t. x*.

(i) $\tan^{-1}\left(\dfrac{8x}{1-15x^2}\right)$ (ii) $\cot^{-1}\left(\dfrac{1+35x^2}{2x}\right)$

(iii) $\tan^{-1}\left(\dfrac{2\sqrt{x}}{1+3x}\right)$ (iv) $\tan^{-1}\left(\dfrac{2^{x+2}}{1-3(4^x)}\right)$

(v) $\tan^{-1}\left(\dfrac{2^x}{1+2^{2x+1}}\right)$ (vi) $\cot^{-1}\left(\dfrac{a^2-6x^2}{5ax}\right)$

(vii) $\tan^{-1}\left(\dfrac{a+b\tan x}{b-a\tan x}\right)$

(viii) $\tan^{-1}\left(\dfrac{5-x}{6x^2-5x-3}\right)$

(ix) $\cot^{-1}\left(\dfrac{4-x-2x^2}{3x+2}\right)$

1.3.1 Logarithmic Differentiation

The complicated functions given by formulas that involve products, quotients and powers can often be simplified more quickly by taking the natural logarithms on both the sides. This enables us to use the laws of logarithms to simplify the functions and differentiate easily. Especially when the functions are of the form $y = [f(x)]^{g(x)}$ it is recommended to take logarithms on both the sides which simplifies to $\log y = g(x).\log[f(x)]$, now it becomes convenient to find the derivative. This process of finding the derivative is called logarithmic differentiation.

SOLVED EXAMPLES

Ex. 1 : Differentiate the following *w. r. t. x*.

(i) $\left(\dfrac{(x^2+3)^2\sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}}\right)$ (ii) $\dfrac{e^{x^2}(\tan x)^{\frac{x}{2}}}{(1+x^2)^{\frac{3}{2}}\cos^3 x}$

(iii) $(x+1)^{\frac{3}{2}}(2x+3)^{\frac{5}{2}}(3x+4)^{\frac{2}{3}}$ for $x \geq 0$ (iv) $x^a + x^x + a^x$ (v) $(\sin x)^{\tan x} - x^{\log x}$

Solution :

(i) Let $y = \left(\dfrac{(x^2+3)^2\sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}}\right)$

Taking log of both the sides we get,

$$\log y = \log\left(\frac{(x^2+3)^2\sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}}\right) = \log\left[\frac{(x^2+3)^2(x^3+5)^{\frac{2}{3}}}{(2x^2+1)^{\frac{3}{2}}}\right]$$

$$= \log\left[(x^2+3)^2(x^3+5)^{\frac{2}{3}}\right] - \log(2x^2+1)^{\frac{3}{2}}$$

$$= \log(x^2+3)^2 + \log(x^3+5)^{\frac{2}{3}}\Big] - \log(2x^2+1)^{\frac{3}{2}}$$

$$\log y = 2\log(x^2+3) + \frac{2}{3}\log(x^3+5) - \frac{3}{2}\log(2x^2+1)$$

Differentiate *w. r. t. x.*

$$\frac{d}{dx}(\log y) = \frac{d}{dx}\left[2\log(x^2+3) + \frac{2}{3}\log(x^3+5) - \frac{3}{2}\log(2x^2+1)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = 2\cdot\frac{d}{dx}[\log(x^2+3)] + \frac{2}{3}\cdot\frac{d}{dx}[\log(x^3+5)] - \frac{3}{2}\cdot\frac{d}{dx}[\log(2x^2+1)]$$

$$= \frac{2}{x^2+3}\cdot\frac{d}{dx}(x^2+3) + \frac{2}{3(x^3+5)}\cdot\frac{d}{dx}(x^3+5) - \frac{3}{2(2x^2+1)}\cdot\frac{d}{dx}(2x^2+1)$$

$$\frac{dy}{dx} = y\left[\frac{2}{x^2+3}(2x) + \frac{2}{3(x^3+5)}(3x^2) - \frac{3}{2(2x^2+1)}(4x)\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x^2+3)^2\sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}}\left[\frac{4x}{x^2+3} + \frac{2x^2}{(x^3+5)} - \frac{6x}{2x^2+1}\right]$$

(ii) Let $y = \dfrac{e^{x^2}(\tan x)^{\frac{x}{2}}}{(1+x^2)^{\frac{3}{2}}\cos^3 x}$

Taking log of both the sides we get,

$$\log y = \log\left(\frac{e^{x^2}(\tan x)^{\frac{x}{2}}}{(1+x^2)^{\frac{3}{2}}(\cos x)^3}\right) = \log\left[e^{x^2}(\tan x)^{\frac{x}{2}}\right] - \log\left[(1+x^2)^{\frac{3}{2}}(\cos x)^3\right]$$

$$= \log e^{x^2} + \log(\tan x)^{\frac{x}{2}} - \left[\log(1+x^2)^{\frac{3}{2}} + \log(\cos x)^3\right]$$

$$= x^2\log e + \frac{x}{2}\log(\tan x) - \frac{3}{2}\log(1+x^2) - 3\log(\cos x)$$

$$\therefore \quad \log y = x^2 + \frac{x}{2}\log(\tan x) - \frac{3}{2}\log(1+x^2) - 3\log(\cos x)$$

Differentiate *w. r. t. x.*

$$\frac{d}{dx}(\log y) = \frac{d}{dx}\left[x^2 + \frac{x}{2}\log(\tan x) - \frac{3}{2}\log(1+x^2) - 3\log(\cos x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{x}{2}\cdot\frac{d}{dx}[\log(\tan x)] + \log(\tan x)\frac{d}{dx}\left(\frac{x}{2}\right) - \frac{3}{2}\frac{d}{dx}[\log(1+x^2)] - 3\frac{d}{dx}[\log(\cos x)]$$

$$= 2x + \frac{x}{2}\cdot\frac{1}{\tan x}\cdot\frac{d}{dx}(\tan x) + \log(\tan x)\frac{1}{2} - \frac{3}{2(1+x^2)}\cdot\frac{d}{dx}(1+x^2) - \frac{3}{\cos x}\cdot\frac{d}{dx}(\cos x)$$

$$= 2x + \frac{x}{2}\cdot(\cot x)(\sec^2 x) + \frac{1}{2}\log(\tan x) - \frac{3}{2(1+x^2)}\cdot(2x) - \frac{3}{\cos x}(-\sin x)$$

$$= 2x + \frac{x}{2}\times\frac{\cos x}{\sin x}\times\frac{1}{\cos^2 x} + \frac{1}{2}\log(\tan x) - \frac{3x}{1+x^2} + 3\tan x$$

$$\frac{dy}{dx} = y\left[2x + \frac{x}{2\sin x\cos x} + \frac{1}{2}\log(\tan x) - \frac{3x}{1+x^2} + 3\tan x\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{e^{x^2}(\tan x)^{\frac{x}{2}}}{(1+x^2)^{\frac{3}{2}}\cos^3 x}\left[2x + x\operatorname{cosec} 2x + \frac{1}{2}\log(\tan x) - \frac{3x}{1+x^2} + 3\tan x\right]$$

(iii) Let $y = (x+1)^{\frac{3}{2}}(2x+3)^{\frac{5}{2}}(3x+4)^{\frac{2}{3}}$

Taking log of both the sides we get,

$$\log y = \log\left[(x+1)^{\frac{3}{2}}(2x+3)^{\frac{5}{2}}(3x+4)^{\frac{2}{3}}\right]$$

$$= \log(x+1)^{\frac{3}{2}} + \log(2x+3)^{\frac{5}{2}} + \log(3x+4)^{\frac{2}{3}}$$

$$\log y = \frac{3}{2}\log(x+1) + \frac{5}{2}\log(2x+3) + \frac{2}{3}\log(3x+4)$$

Differentiate *w. r. t. x.*

$$\frac{d}{dx}(\log y) = \frac{d}{dx}\left[\frac{3}{2}\log(x+1) + \frac{5}{2}\log(2x+3) + \frac{2}{3}\log(3x+4)\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{3}{2}\cdot\frac{d}{dx}[\log(x+1)] + \frac{5}{2}\cdot\frac{d}{dx}[\log(3x+2)] + \frac{2}{3}\cdot\frac{d}{dx}[\log(3x+4)]$$

$$= \frac{3}{2(x+1)}\cdot\frac{d}{dx}(x+1) + \frac{5}{2(3x+1)}\cdot\frac{d}{dx}(3x+2) + \frac{2}{3(3x+4)}\cdot\frac{d}{dx}(3x+4)$$

$$\frac{dy}{dx} = y\left[\frac{3}{2(x+1)} + \frac{5}{2(3x+1)}(3) + \frac{2}{3(3x+4)}(3)\right]$$

$$\therefore \quad \frac{dy}{dx} = (x+1)^{\frac{3}{2}}(2x+3)^{\frac{5}{2}}(3x+4)^{\frac{2}{3}}\left[\frac{3}{2(x+1)} + \frac{15}{2(3x+1)} + \frac{2}{3x+4}\right]$$

(iv) Let $y = x^a + x^x + a^x$

Here the derivatives of x^a and a^x can be found directly but we can not find the derivative of x^x without the use of logarithm. So the given function is split in to two functions, find their derivatives and then add them.

Let $u = x^a + a^x$ and $v = x^x$

$\therefore \quad y = u + v$, where u and v are differentiable functions of x.

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots\ldots \text{(I)}$$

Now, $u = x^a + a^x$

Differentiate *w. r. t. x.*

$$\frac{du}{dx} = \frac{d}{dx}(x^a) + \frac{d}{dx}(a^x)$$

$$\frac{du}{dx} = ax^{a-1} + a^x \log a \qquad \ldots\ldots \text{(II)}$$

And, $v = x^x$

Taking log of both the sides we get,

$$\log v = \log x^x$$

$$\log v = x \log x$$

Differentiate *w. r. t. x.*

$$\frac{d}{dx}(\log v) = \frac{d}{dx}(x \log x)$$

$$\frac{1}{v}\frac{dv}{dx} = x\frac{d}{dx}(\log x) + \log x \frac{d}{dx}(x)$$

$$\frac{dv}{dx} = v\left[x \times \frac{1}{x} + \log x\,(1)\right]$$

$$\frac{dv}{dx} = x^x\,[1 + \log x] \qquad \ldots\ldots \text{(III)}$$

Substituting (II) and (III) in (I) we get,

$$\frac{dy}{dx} = ax^{a-1} + a^x \log a + x^x\,[1 + \log x]$$

(v) Let $y = (\sin x)^{\tan x} - x^{\log x}$

Let $u = (\sin x)^{\tan x}$ and $v = x^{\log x}$

$\therefore \quad y = u - v$, where u and v are differentiable functions of x.

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \qquad \ldots\ldots \text{(I)}$$

Now, $u = (\sin x)^{\tan x}$, taking log of both the sides we get,

$$\log u = \log (\sin x)^{\tan x} \qquad \therefore \qquad \log u = \tan x \log (\sin x)$$

Differentiate *w. r. t. x.*

$$\frac{d}{dx}(\log u) = \frac{d}{dx}[\tan x \log (\sin x)]$$

$$\frac{1}{u}\frac{du}{dx} = \tan x \frac{d}{dx}[\log (\sin x)] + \log (\sin x) \frac{d}{dx}(\tan x)$$

$$= \tan x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx} \cdot (\sin x) + \log (\sin x) \cdot (\sec^2)$$

$$\frac{du}{dx} = u\left[\tan x \cdot \frac{1}{\sin x} \cdot (\cos x) + \sec^2 x \cdot \log(\sin x)\right]$$

$$\frac{du}{dx} = (\sin x)^{\tan x}[\tan x \cdot \cot x + \sec^2 x \cdot \log(\sin x)]$$

$$\frac{du}{dx} = (\sin x)^{\tan x}[1 + \sec^2 x \cdot \log(\sin x)] \qquad \ldots\ldots \text{(II)}$$

And, $\quad v = x^{\log x}$

Taking log on both the sides we get,

$$\log v = \log\left(x^{\log x}\right)$$

$$\log v = \log x \log x = (\log x)^2$$

Differentiate *w. r. t. x.*

$$\frac{d}{dx}(\log v) = \frac{d}{dx}[(\log x)^2]$$

$$\frac{1}{v}\frac{dv}{dx} = 2\log x \frac{d}{dx}(\log x)$$

$$\frac{dv}{dx} = u\left[\frac{2\log x}{x}\right] = \frac{2x^{\log x}\log x}{x} \qquad \ldots\ldots \text{(III)}$$

Substituting (II) and (III) in (I) we get,

$$\frac{dy}{dx} = (\sin x)^{\tan x}[1 + \sec^2 x \cdot \log(\sin x)] - \frac{2x^{\log x}\log x}{x}$$

1.3.2 Implicit Functions

Functions can be represented in a variety of ways. Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y solely in terms of the variable x. It is not always possible to solve for one variable explicitly in terms of another. Those cases where it is possible to solve for one variable in terms of another to obtain $y = f(x)$ or $x = g(y)$ are said to be in **explicit** form.

If an equation in x and y is given but x is not an explicit function of y and y is not an explicit function of x then either of the variables is an **Implicit function** of the other.

1.3.3 Derivatives of Implicit Functions

1. Differentiate both sides of the equation with respect to x (independent variable), treating y as a differentiable function of x.

2. Collect the terms containing $\frac{dy}{dx}$ on one side of the equation and solve for $\frac{dy}{dx}$.

SOLVED EXAMPLES

Ex. 1 : Find $\frac{dy}{dx}$ if

(i) $x^5 + xy^3 + x^2y + y^4 = 4$

(ii) $y^3 + \cos(xy) = x^2 - \sin(x + y)$

(iii) $x^2 + e^{xy} = y^2 + \log(x + y)$

Solution :

(i) Given that : $x^5 + xy^3 + x^2y + y^4 = 4$

Differentiate *w. r. t. x.*

$$\frac{d}{dx}(x^5) + \frac{d}{dx}(xy^3) + \frac{d}{dx}(x^2y) + \frac{d}{dx}(y^4) = \frac{d}{dx}(4)$$

$$5x^4 + x\frac{d}{dx}(y^3) + y^3\frac{d}{dx}(x) + x^2\frac{d}{dx}(y) + y\frac{d}{dx}(x^2) + 4y^3\frac{d}{dx}(y) = 0$$

$$5x^4 + x(3y^2)\frac{dy}{dx} + y^3(1) + x^2\frac{dy}{dx} + y(2x) + 4y^3\frac{dy}{dx} = 0$$

$$x^2\frac{dy}{dx} + 3xy^2\frac{dy}{dx} + 4y^3\frac{dy}{dx} = -5x^4 - 2xy - y^3$$

$$(x^2 + 3xy^2 + 4y^3)\frac{dy}{dx} = -(5x^4 + 2xy + y^3)$$

$$\therefore \quad \frac{dy}{dx} = -\frac{5x^4 + 2xy + y^3}{x^2 + 3xy^2 + 4y^3}$$

(ii) Given that : $y^3 + \cos(xy) = x^2 - \sin(x + y)$

Differentiate *w. r. t. x.*

$$\frac{d}{dx}(y^3) + \frac{d}{dx}[\cos(xy)] = \frac{d}{dx}(x^2) - \frac{d}{dx}[\sin(x + y)]$$

$$3y^2\frac{d}{dx}(y) - \sin(xy)\frac{d}{dx}(xy) = 2x - \cos(x + y)\frac{d}{dx}(x + y)$$

$$3y^2\frac{dy}{dx} - \sin(xy)\left[x\frac{dy}{dx} + y(1)\right] = 2x - \cos(x + y)\left[1 + \frac{dy}{dx}\right]$$

$$3y^2\frac{dy}{dx} - x\sin(xy)\frac{dy}{dx} - y\sin(xy) = 2x - \cos(x + y) - \cos(x + y)\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - x\sin(xy)\frac{dy}{dx} + \cos(x + y)\frac{dy}{dx} = 2x + y\sin(xy) - \cos(x + y)$$

$$[3y^2 - x\sin(xy) + \cos(x + y)]\frac{dy}{dx} = 2x + y\sin(xy) - \cos(x + y)$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x + y\sin(xy) - \cos(x + y)}{3y^2 - x\sin(xy) + \cos(x + y)}$$

(iii) Given that : $x^2 + e^{xy} = y^2 + \log(x + y)$

Recall that $\frac{d}{dx} g(f(x)) = g'(f(x)) \cdot \frac{d}{dx} f(x)$

Differentiate $w.\ r.\ t.\ x.$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}[e^{xy}] = \frac{d}{dx}(y^2) + \frac{d}{dx}[\log(x + y)]$$

$$2x + e^{xy}\frac{d}{dx}(xy) = 2y\frac{dy}{dx} + \frac{1}{x+y}\frac{d}{dx}(x + y)$$

$$2x + e^{xy}\left[x\frac{dy}{dx} + y(1)\right] = 2y\frac{dy}{dx} + \frac{1}{x+y}\left[1 + \frac{dy}{dx}\right]$$

$$2x + xe^{xy}\frac{dy}{dx} + ye^{xy} = 2y\frac{dy}{dx} + \frac{1}{x+y} + \frac{1}{x+y}\cdot\frac{dy}{dx}$$

$$2x + ye^{xy} - \frac{1}{x+y} = 2y\frac{dy}{dx} - xe^{xy}\frac{dy}{dx} + \frac{1}{x+y}\cdot\frac{dy}{dx}$$

$$2x + ye^{xy} - \frac{1}{x+y} = \left[2y - xe^{xy} + \frac{1}{x+y}\right]\frac{dy}{dx}$$

$$\frac{2x(x+y) + ye^{xy}(x+y) - 1}{x+y} = \left[\frac{2y(x+y) - xe^{xy}(x+y) + 1}{x+y}\right]\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{2x(x+y) + ye^{xy}(x+y) - 1}{2y(x+y) - xe^{xy}(x+y) + 1}$$

Ex. 2 : Find $x^m \cdot y^n = (x + y)^{m+n}$, then prove that $\frac{dy}{dx} = \frac{y}{x}$.

Solution : Given that : $x^m \cdot y^n = (x + y)^{m+n}$

Taking log on both the sides, we get

$$\log[x^m \cdot y^n] = \log[(x + y)^{m+n}]$$

$$m\log x + n\log y = (m + n)\log(x + y)$$

Differentiate $w.\ r.\ t.\ x.$

$$m\frac{d}{dx}(\log x) + n\frac{d}{dx}(\log y) = (m + n)\frac{d}{dx}[\log(x + y)]$$

$$\frac{m}{x} + \frac{n}{y}\cdot\frac{dy}{dx} = \frac{m+n}{x+y}\cdot\frac{d}{dx}(x + y)$$

$$\frac{m}{x} + \frac{n}{y}\cdot\frac{dy}{dx} = \frac{m+n}{x+y}\cdot\left[1 + \frac{dy}{dx}\right]$$

$$\frac{m}{x} + \frac{n}{y}\cdot\frac{dy}{dx} = \frac{m+n}{x+y} + \frac{m+n}{x+y}\cdot\frac{dy}{dx}$$

$$\frac{n}{y}\cdot\frac{dy}{dx} - \frac{m+n}{x+y}\cdot\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\left[\frac{n}{y} - \frac{m+n}{x+y}\right]\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\left[\frac{n(x+y) - (m+n)y}{y(x+y)}\right]\frac{dy}{dx} = \left[\frac{(m+n)x - m(x+y)}{x(x+y)}\right]$$

$$\left[\frac{nx + ny - my - ny}{y}\right]\frac{dy}{dx} = \frac{mx + nx - mx - my}{x}$$

$$\left[\frac{nx - my}{y}\right]\frac{dy}{dx} = \frac{nx - my}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x}$$

Ex. 3 : If $\sin\left(\frac{px^m - qy^m}{px^m + qy^m}\right) = r$, then show that $\frac{dy}{dx} = \frac{y}{x}$, where r is a constant.

Solution : Given that : $\sin\left(\frac{px^m - qy^m}{px^m + qy^m}\right) = r$

$$\frac{px^m - qy^m}{px^m + qy^m} = \sin^{-1} r$$

$$\frac{px^m - qy^m}{px^m + qy^m} = t \qquad \text{...... [Let } t = \sin^{-1} r \text{]}$$

$$px^m - qy^m = ptx^m + qty^m$$

$$px^m - ptx^m = qy^m + qty^m$$

$$p(1-t)x^m = q(1+t)y^m$$

$$y^m = \left(\frac{p(1-t)}{q(1+t)}\right)x^m$$

$$y^m = s \cdot x^m \qquad \text{...... (I)} \qquad \text{......} \left[\text{Let } s = \left(\frac{p(1-t)}{q(1-t)}\right)\right]$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}(y^m) = s\frac{d}{dx}(x^m)$$

$$my^{m-1}\frac{dy}{dx} = s \cdot mx^{m-1}$$

$$\frac{dy}{dx} = s \cdot \frac{x^{m-1}}{y^{m-1}} x^{m-1}$$

$$\frac{dy}{dx} = \frac{y^m}{x^m} \times \frac{x^{m-1}}{y^{m-1}} \qquad \text{...... [From (I)]}$$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x}$$

Ex. 4 : If $\sec^{-1}\left(\dfrac{x^3+y^3}{x^3-y^3}\right)=2a$, then show that $\dfrac{dy}{dx}=\dfrac{x^2\tan^2 a}{y^2}$, where a is a constant.

Solution : Given that : $\sec^{-1}\left(\dfrac{x^3+y^3}{x^3-y^3}\right)=2a$ . . . . . [We will not eliminate a, as answer contains a]

$$\therefore \quad \cos^{-1}\left(\frac{x^3-y^3}{x^3+y^3}\right)=2a$$

$$\frac{x^3-y^3}{x^3+y^3}=\cos 2a$$

$$x^3-y^3=x^3\cos 2a+y^3\cos 2a$$

$$x^3-x^3\cos 2a=y^3\cos 2a+y^3$$

$$x^3(1-\cos 2a)=y^3(1+\cos 2a)$$

$$y^3=\left(\frac{1-\cos 2a}{1+\cos 2a}\right)x^3$$

$$y^3=\left(\frac{2\sin^2 a}{2\cos^2 a}\right)x^3$$

$$y^3=(\tan^2 a)\,x^3 \qquad \text{. . . . . . (I)}$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}(y^3)=(\tan^2 a)\frac{d}{dx}(x^3)$$

$$3y^2\frac{dy}{dx}=(\tan^2 a)\,3x^2$$

$$\therefore \quad \frac{dy}{dx}=\frac{x^2\tan^2 a}{y^2}$$

Ex. 5 : If $y=\sqrt{\tan x+\sqrt{\tan x+\sqrt{\tan x+\ldots\infty}}}$, then show that $\dfrac{dy}{dx}=\dfrac{\sec^2 x}{2y-1}$.

Solution : Given that : $y=\sqrt{\tan x+\sqrt{\tan x+\sqrt{\tan x+\ldots\infty}}}$ . . . . (I)

Squaring both sides, we get

$y^2 = \tan x+\sqrt{\tan x+\sqrt{\tan x+\ldots\infty}}$, which is same as

$$y^2 = \tan x+\sqrt{\tan x+\sqrt{\tan x+\sqrt{\tan x+\ldots\infty}}}$$

$$y^2 = \tan x+y \qquad \text{. . . . . . . [From (I)]}$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}(y^2)=\frac{d}{dx}(\tan x)+\frac{dy}{dx}$$

$$2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$$

$$(2y - 1)\frac{dy}{dx} = \sec^2 x$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sec^2 x}{2y - 1}$$

Ex. 6 : If $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$, then show that $\frac{dy}{dx} = \sqrt{\frac{1 - y^2}{1 - x^2}}$.

Solution : Given that : $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$ .... (I)

Put $x = \sin\alpha$, $y = \sin\beta$

$\therefore \quad \alpha = \sin^{-1} x$, $\beta = \sin^{-1} y$

Equation (I) becomes,

$$\sqrt{1 - \sin^2\alpha} + \sqrt{1 - \sin^2\beta} = a(\sin\alpha - \sin\beta)$$

$$\cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$$

$$2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) = 2a\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos\left(\frac{\alpha - \beta}{2}\right) = a\sin\left(\frac{\alpha + \beta}{2}\right) \Rightarrow \cot\left(\frac{\alpha - \beta}{2}\right) = a$$

$$\frac{\alpha - \beta}{2} = \cot^{-1} a \qquad \therefore \qquad \alpha - \beta = 2\cot^{-1} a$$

$$\therefore \quad \sin^{-1} x - \sin^{-1} y = 2\cot^{-1} a$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}(\sin^{-1} x) - \frac{d}{dx}(\sin^{-1} y) = \frac{d}{dx}(2\cot^{-1} a)$$

$$\frac{1}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 - y^2}} \cdot \frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \sqrt{\frac{1 - y^2}{1 - x^2}}$$

EXERCISE 1.3

(1) Differentiate the following *w. r. t. x*

(i) $\frac{(x + 1)^2}{(x + 2)^3 (x + 3)^4}$

(ii) $\sqrt[3]{\frac{4x - 1}{(2x + 3)(5 - 2x)^2}}$

(iii) $(x^2 + 3)^{\frac{3}{2}} \cdot \sin^3 2x \cdot 2^{x^2}$

(iv) $\frac{(x^2 + 2x + 2)^{\frac{3}{2}}}{(\sqrt{x} + 3)^3 (\cos x)^x}$

(v) $\frac{x^5 \cdot \tan^3 4x}{\sin^2 3x}$

(vi) $x^{\tan^{-1} x}$

(vii) $(\sin x)^x$

(viii) $\sin x^x$

(2) Differentiate the following *w. r. t. x*.

(i) $x^e + x^x + e^x + e^e$ (ii) $x^{x^x} + e^{x^x}$

(iii) $(\log x)^x - (\cos x)^{\cot x}$

(iv) $x^{e^x} + (\log x)^{\sin x}$ (v) $e^{\tan x} + (\log x)^{\tan x}$

(vi) $(\sin x)^{\tan x} + (\cos x)^{\cot x}$

(vii) $10^{x^x} + x^{x^{10}} + x^{10^x}$

(viii) $[(\tan x)^{\tan x}]^{\tan x}$ at $x = \dfrac{\pi}{4}$

(3) Find $\dfrac{dy}{dx}$ if

(i) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ (ii) $x\sqrt{x} + y\sqrt{y} = a\sqrt{a}$

(iii) $x + \sqrt{xy} + y = 1$

(iv) $x^3 + x^2 y + xy^2 + y^3 = 81$

(v) $x^2 y^2 - \tan^{-1}\sqrt{x^2 + y^2} = \cot^{-1}\sqrt{x^2 + y^2}$

(vi) $xe^y + ye^x = 1$ (vii) $e^{x+y} = \cos(x - y)$

(viii) $\cos(xy) = x + y$ (ix) $e^{e^{x-y}} = \dfrac{x}{y}$

(x) $x + \sin(x + y) = y - \cos(x - y)$

(4) Show that $\dfrac{dy}{dx} = \dfrac{y}{x}$ in the following, where a and p are constants.

(i) $x^7 y^5 = (x + y)^{12}$

(ii) $x^p y^4 = (x + y)^{p+4}$, $p \in N$

(iii) $\sec\left(\dfrac{x^5 + y^5}{x^5 - y^5}\right) = a^2$

(iv) $\tan^{-1}\left(\dfrac{3x^2 - 4y^2}{3x^2 + 4y^2}\right) = a^2$

(v) $\cos^{-1}\left(\dfrac{7x^4 + 5y^4}{7x^4 - 5y^4}\right) = \tan^{-1} a$

(vi) $\log\left(\dfrac{x^{20} - y^{20}}{x^{20} + y^{20}}\right) = 20$

(vii) $e^{\frac{x^7 - y^7}{x^7 + y^7}} = a$

(viii) $\sin\left(\dfrac{x^3 - y^3}{x^3 + y^3}\right) = a^3$

(5) (i) If $\log(x + y) = \log(xy) + p$, where p is constant then prove that $\dfrac{dy}{dx} = -\dfrac{y^2}{x^2}$.

(ii) If $\log_{10}\left(\dfrac{x^3 - y^3}{x^3 + y^3}\right) = 2$, show that $\dfrac{dy}{dx} = -\dfrac{99x^2}{101y^2}$.

(iii) If $\log_5\left(\dfrac{x^4 + y^4}{x^4 - y^4}\right) = 2$, show that $\dfrac{dy}{dx} = -\dfrac{12x^3}{13y^3}$.

(iv) If $e^x + e^y = e^{x+y}$, then show that $\dfrac{dy}{dx} = -e^{y-x}$.

(v) If $\sin^{-1}\left(\dfrac{x^5 - y^5}{x^5 + y^5}\right) = \dfrac{\pi}{6}$, show that $\dfrac{dy}{dx} = \dfrac{x^4}{3y^4}$.

(vi) If $x^y = e^{x-y}$, then show that $\dfrac{dy}{dx} = \dfrac{\log x}{(1 + \log x)^2}$.

(vii) If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots \infty}}}$, then show that $\dfrac{dy}{dx} = \dfrac{\sin x}{1 - 2y}$.

(viii) If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots \infty}}}$, then show that $\dfrac{dy}{dx} = \dfrac{1}{x(2y - 1)}$.

(ix) If $y = x^{x^{x^{\cdot^{\cdot^{\cdot^{\infty}}}}}}$, then show that $\dfrac{dy}{dx} = \dfrac{y^2}{x(1 - \log y)}$.

(x) If $e^y = y^x$, then show that $\dfrac{dy}{dx} = \dfrac{(\log y)^2}{\log y - 1}$.

1.4.1 Derivatives of Parametric Functions

Consider the equations $x = f(t), y = g(t)$. These equations may imply a functional relation between the variables x and y. Given the value of t in some domain $[a, b]$, we can find x and y.

For example $x = a \cos t$ and $y = a \sin t$. The functional relation between these two functions is that, $x^2 + y^2 = a^2 \cos^2 t + a^2 \sin^2 t = a^2 (\cos^2 t + \sin^2 t) = a^2$ represents the equation of a circle of radius a with center at the origin. And the domain of t is $[0, 2\pi]$. We can find x and y for any $t \in [0, 2\pi]$.

If two variables x and y are defined separately as functions by an inter mediating varibale t, then that inter mediating variable is known as parameter. Let us discuss the derivatives of parametric functions.

1.4.2 Theorem : If $x = f(t)$ and $y = g(t)$ are differentiable functions of t so that y is a differentiable function of x and if $\frac{dx}{dt} \neq 0$ then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Proof : Given that $x = f(t)$ and $y = g(t)$.

Let there be a small increment in the value of t say δt then δx and δy are the corresponding increments in x and y respectively.

As $\delta t, \delta x, \delta y$ are small increments in t, x and y respectively such that $\delta t \neq 0$ and $\delta x \neq 0$.

Consider, the incrementary ratio $\frac{\delta y}{\delta x}$, and note that $\delta x \to 0 \Rightarrow \delta t \to 0$.

i.e. $\frac{\delta y}{\delta x} = \frac{\frac{\delta y}{\delta t}}{\frac{\delta x}{\delta t}}$, since $\frac{\delta x}{\delta t} \neq 0$

Taking the limit as $\delta t \to 0$ on both sides we get,

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta t \to 0}\left(\frac{\frac{\delta y}{\delta t}}{\frac{\delta x}{\delta t}}\right)$$

As $\delta t \to 0, \delta x \to 0$

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{\lim_{\delta t \to 0}\left(\frac{\delta y}{\delta t}\right)}{\lim_{\delta t \to 0}\left(\frac{\delta x}{\delta t}\right)} \qquad \text{.....(I)}$$

Since x and y are differentiable function of t. we have,

$$\lim_{\delta t \to 0}\left(\frac{\delta x}{\delta t}\right) = \frac{dx}{dt} \text{ and } \lim_{\delta t \to 0}\left(\frac{\delta y}{\delta t}\right) = \frac{dy}{dt} \text{ exist and are finite} \qquad \text{.....(II)}$$

From (I) and (II), we get

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{.....(III)}$$

The R.H.S. of (III) exists and is finite, implies L.H.S.of (III) also exist and finite

$$\therefore \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{dy}{dx}$$

Thus the equation (III) becomes,

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}} \qquad \text{where } \frac{dx}{dt} \neq 0$$

SOLVED EXAMPLES

Ex. 1 : Find $\frac{dy}{dx}$ if

(i) $x = at^4, y = 2at^2$ (ii) $x = t - \sqrt{t}, y = t + \sqrt{t}$

(iii) $x = \cos(\log t), \ y = \log(\cos t)$ (iv) $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$

(v) $x = \sqrt{1 - t^2}, \ y = \sin^{-1} t$

Solution :

(i) Given, $y = 2at^2$

Differentiate *w. r. t. t*

$$\frac{dy}{dt} = 2a \frac{d}{dt}(t^2) = 2a(2t) = 4at \quad \ldots\ldots (I)$$

And, $x = at^4$

Differentiate *w. r. t. t*

$$\frac{dx}{dt} = a \frac{d}{dt}(t^4) = a(4t^3) = 4at^3 \quad \ldots\ldots (II)$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4at}{4at^3}$ ...[From (I) and (II)]

$\therefore \quad \frac{dy}{dx} = \frac{1}{t^2}$

(ii) Given, $y = t + \sqrt{t}$

Differentiate *w. r. t. t*

$$\frac{dy}{dt} = \frac{d}{dt}(t + \sqrt{t}) = 1 + \frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dt} = \frac{2\sqrt{t} + 1}{2\sqrt{t}} \quad \ldots\ldots (I)$$

And, $x = t - \sqrt{t}$

Differentiate *w. r. t. t*

$$\frac{dx}{dt} = \frac{d}{dt}(t - \sqrt{t}) = 1 - \frac{1}{2\sqrt{t}}$$

$$\frac{dx}{dt} = \frac{2\sqrt{t} - 1}{2\sqrt{t}} \quad \ldots\ldots (II)$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2\sqrt{t}+1}{2\sqrt{t}}}{\frac{2\sqrt{t}-1}{2\sqrt{t}}}$ ...[From (I) and (II)]

$\therefore \quad \frac{dy}{dx} = \frac{2\sqrt{t} + 1}{2\sqrt{t} - 1}$

(iii) Given, $y = \log(\cos t)$

Differentiate *w. r. t. t*

$$\frac{dy}{dt} = \frac{d}{dt}[\log(\cos t)] = \frac{1}{\cot t} \cdot \frac{d}{dt}(\cos t) = \frac{1}{\cot t}(-\sin t) \qquad \therefore \frac{dy}{dt} = -\tan t \quad \ldots\ldots (I)$$

And, $x = \cos(\log t)$

Differentiate *w. r. t. t*

$$\frac{dx}{dt} = \frac{d}{dt}[\cos(\log t)] = -\sin(\log t) \cdot \frac{d}{dt}(\log t) = -\frac{\sin(\log t)}{t} \qquad \therefore \frac{dx}{dt} = -\frac{\sin(\log t)}{t} \quad \ldots\ldots (II)$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\tan t}{-\frac{\sin(\log t)}{t}}$ ...[From (I) and (II)]

$\therefore \quad \frac{dy}{dx} = \frac{t \cdot \tan t}{\sin(\log t)}$

(iv) Given, $y = a(1 - \cos\theta)$

Differentiate $w.\ r.\ t.\ \theta$

$$\frac{dy}{d\theta} = a\frac{d}{d\theta}[(1-\cos\theta)] = a[0-(-\sin\theta)]$$

$$\frac{dy}{dt} = a\sin\theta \qquad \text{.....(I)}$$

And, $x = a(\theta + \sin\theta)$

Differentiate $w.\ r.\ t.\ \theta$

$$\frac{dx}{dt} = a\frac{d}{d\theta}(\theta+\sin\theta) = a(1+\cos\theta)$$

$$\frac{dx}{dt} = a(1+\cos\theta) \qquad \text{.....(II)}$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\sin\theta}{a(1+\cos\theta)} \qquad \text{...[From (I) and (II)]}$$

$$\therefore \quad \frac{dy}{dx} = \frac{2\sin\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)} = \tan\left(\frac{\theta}{2}\right)$$

(v) Given, $y = \sin^{-1} t$

Differentiate $w.\ r.\ t.\ t$

$$\frac{dy}{dt} = \frac{d}{dt}(\sin^{-1} t) = \frac{1}{\sqrt{1-t^2}}$$

$$\frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}} \qquad \text{.....(I)}$$

And, $x = \sqrt{1-t^2}$

Differentiate $w.\ r.\ t.\ t$

$$\frac{dx}{dt} = \frac{d}{dt}\left(\sqrt{1-t^2}\right) = \frac{1}{2\sqrt{1-t^2}}\cdot\frac{d}{dt}(1-t^2)$$

$$\frac{dx}{dt} = \frac{1}{2\sqrt{1-t^2}}\cdot(-2t) = -\frac{t}{\sqrt{1-t^2}} \qquad \text{.....(II)}$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{\sqrt{1-t^2}}}{-\frac{t}{\sqrt{1-t^2}}} \qquad \text{... [From (I) and (II)]}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{t}$$

Ex. 2 : Find $\frac{dy}{dx}$ if (i) $x = \sec^2\theta,\ y = \tan^3\theta$, at $\theta = \frac{\pi}{3}$ (ii) $x = t + \frac{1}{t},\ y = \frac{1}{t^2}$, at $t = \frac{1}{2}$

(iii) $x = 3\cos t - 2\cos^3 t,\ y = 3\sin t - 2\sin^3 t$, at $t = \frac{\pi}{6}$

Solution :

(i) Given, $y = \tan^3\theta$

Differentiate $w.\ r.\ t.\ \theta$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\tan\theta)^3 = 3\tan^2\theta\frac{d}{d\theta}(\tan\theta) \qquad \therefore \quad \frac{dy}{d\theta} = 3\tan^2\theta\cdot\sec^2\theta \qquad \text{.....(I)}$$

And, $x = \sec^2\theta$

Differentiate $w.\ r.\ t.\ \theta$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\sec^2\theta) = 2\sec\theta\cdot\frac{d}{d\theta}(\sec\theta)$$

$$\frac{dx}{d\theta} = 2\sec\theta\cdot\sec\theta\tan\theta = 2\sec^2\theta\cdot\tan\theta \qquad \text{.....(II)}$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\tan^2\theta\cdot\sec^2\theta}{2\sec^2\theta\cdot\tan\theta} \qquad \text{...[From (I) and (II)]}$$

$$\therefore \quad \frac{dy}{dx} = \frac{3}{2}\tan\theta$$

At $\theta = \frac{\pi}{3}$, we get

$$\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{3}} = \frac{3}{2}\tan\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$$

(ii) Given, $y = \frac{1}{t^2}$

Differentiate *w. r. t. t*

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{1}{t^2}\right)$$

$$\frac{dy}{dt} = -\frac{2}{t^3} \qquad \ldots\ldots (I)$$

And, $x = t + \frac{1}{t}$

Differentiate *w. r. t. t*

$$\frac{dx}{dt} = \frac{d}{dt}\left(t + \frac{1}{t}\right) = 1 - \frac{1}{t^2}$$

$$\frac{dx}{dt} = -\frac{t^2 - 1}{t^2} \qquad \ldots\ldots (II)$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{2}{t^3}}{\frac{t^2-1}{t^2}} \qquad \ldots \text{[From (I) and (II)]}$$

$\therefore$ $$\frac{dy}{dx} = -\frac{2}{t(t^2-1)}$$

At $t = \frac{1}{2}$, we get

$$\left(\frac{dy}{dx}\right)_{t=\frac{1}{2}} = -\frac{2}{\left(\frac{1}{2}\right)\left[\left(\frac{1}{2}\right)^2 - 1\right]}$$

$$= -\frac{2}{\left(\frac{1}{2}\right)\left(\frac{1}{4} - 1\right)}$$

$$\left(\frac{dy}{dx}\right)_{t=\frac{1}{2}} = -\frac{2}{\left(\frac{1}{2}\right)\left(-\frac{3}{4}\right)}$$

$$\left(\frac{dy}{dx}\right)_{t=\frac{1}{2}} = \frac{16}{3}$$

(iii) Given, $y = 3\sin t - 2\sin^3 t$

Differentiate *w. r. t. t*

$$\frac{dy}{dt} = \frac{d}{dt}(3\sin t - 2\sin^3 t)$$

$$= 3\frac{d}{dt}(\sin t) - 2(\sin t)^3$$

$$= 3\cos t - 2(3)\sin^2 t\frac{d}{dt}(\sin t)$$

$$= 3\cos t - 6\sin^2 t(\cos t)$$

$$= 3\cos t(1 - 2\sin^2 t)$$

$$\frac{dy}{dt} = 3\cos t\cos 2t \qquad \ldots\ldots (I)$$

And, $x = 3\cos t - 2\cos^3 t$

Differentiate *w. r. t. t*

$$\frac{dx}{dt} = \frac{d}{dt}(3\cos t - 2\cos^3 t)$$

$$= 3\frac{d}{dt}(\cos t) - 2\frac{d}{dt}(\cos^3 t)$$

$$= 3(-\sin t) - 2(3)\cos^2 t\frac{d}{dt}(\cos t)$$

$$= -3\sin t - 6\cos^2 t(-\sin t)$$

$$= 6\cos^2 t\sin t - 3\sin t$$

$$= 3\sin t(2\cos^2 t - 1)$$

$$\frac{dx}{dt} = 3\sin t\cos 2t \qquad \ldots\ldots (II)$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos t\cos 2t}{3\sin t\cos 2t} \qquad \ldots\ldots \text{[From (I) and (II)]}$$

$\therefore$ $$\frac{dy}{dx} = -\cot t$$

At $t = \frac{\pi}{6}$, we get

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{6}} = -\cot\left(\frac{\pi}{6}\right) = \sqrt{3}$$

Ex. 3 : If $x^2 + y^2 = t + \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$, then show that $x^3y\frac{dy}{dx} = -1$.

Solution :

Given that, $x^4 + y^4 = t^2 + \frac{1}{t^2}$ . . . (I)

And $x^2 + y^2 = t + \frac{1}{t}$

Squaring both sides,

$$(x^2 + y^2)^2 = \left(t + \frac{1}{t}\right)^2$$

$$x^4 + 2x^2y^2 + y^4 = t^2 + 2 + \frac{1}{t^2}$$

$$x^4 + 2x^2y^2 + y^4 = x^4 + y^4 + 2 \quad \text{. . . [From (I)]}$$

$$2x^2y^2 = 2 \quad \therefore \quad x^2y^2 = 1 \quad \text{. . . (II)}$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}(x^2y^2) = \frac{d}{dx}(1)$$

$$x^2\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x^2) = 0$$

$$x^2(2y)\frac{dy}{dx} + y^2(2x) = 0$$

$$2x^2y\frac{dy}{dx} = -2xy^2 \Rightarrow \frac{dy}{dx} = -\frac{2xy^2}{2x^2y}$$

$$\frac{dy}{dx} = -\frac{x\left(\frac{1}{x^2}\right)}{x^2y} \quad \text{. . . [From (II)]}$$

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{x^3y} \quad \therefore \quad x^3y\frac{dy}{dx} = -1$$

Ex. 4 : If $x = a\left(t - \frac{1}{t}\right)$ and $y = b\left(t + \frac{1}{t}\right)$, then show that $\frac{dy}{dx} = \frac{b^2x}{a^2y}$.

Solution :

Given that, $x = a\left(t - \frac{1}{t}\right)$ and $y = b\left(t + \frac{1}{t}\right)$

i.e. $\frac{x}{a} = t - \frac{1}{t}$ . . . (I) and $\frac{y}{b} = t + \frac{1}{t}$ . . . (II)

Square of (I) − Square of (II) gives,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \left(t - \frac{1}{t}\right)^2 - \left(t + \frac{1}{t}\right)^2$$

$$= t^2 - 2 + \frac{1}{t^2} - t^2 - 2 - \frac{1}{t^2}$$

$$\therefore \quad \frac{x^2}{a^2} - \frac{y^2}{a^2} = -4$$

Differentiate *w. r. t. x*

$$\frac{1}{a^2}\cdot\frac{d}{dx}(x^2) - \frac{1}{b^2}\cdot\frac{d}{dx}(y^2) = \frac{d}{dx}(-4)$$

$$\frac{1}{a^2}(2x) - \frac{1}{b^2}(2y)\cdot\frac{d}{dx} = 0$$

$$\frac{1}{a^2}(2x) - \frac{1}{b^2}(2y)\cdot\frac{dy}{dx} = 0$$

$$\frac{2y}{b^2}\cdot\frac{dy}{dx} = \frac{2x}{a^2} \Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

$$\therefore \quad \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

Ex. 5 : If $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$, then show that $\frac{dy}{dx} = -\frac{y}{x}$.

Solution : Given that, $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$

i.e. $x = \sqrt{a^{\sin^{-1}t}}$ . . . (I) and $y = \sqrt{a^{\cos^{-1}t}}$ . . . (II)

Differentiate (I) *w. r. t. t*

$$\frac{dx}{dt} = \frac{d}{dt}\left(\sqrt{a^{\sin^{-1}t}}\right) = \frac{1}{2\sqrt{a^{\sin^{-1}t}}}\cdot\frac{d}{dt}\left(a^{\sin^{-1}t}\right)$$

$$= \frac{1}{2\sqrt{a^{\sin^{-1}t}}}\cdot a^{\sin^{-1}t}\cdot\log a\,\frac{d}{dt}(\sin^{-1}t)$$

$$= \frac{a^{\sin^{-1} t} \cdot \log a}{2\sqrt{a^{\sin^{-1} t}}} \cdot \frac{1}{\sqrt{1-t^2}}$$

$$\frac{dx}{dt} = \frac{\sqrt{a^{\sin^{-1} t}} \cdot \log a}{2\sqrt{1-t^2}} = \frac{x \log a}{2\sqrt{1-t^2}} \quad \ldots \text{(III)} \ldots \text{[From (I)]}$$

Now $y = \sqrt{a^{\cos^{-1} t}}$

Differentiate (II) *w. r. t. t*

$$\frac{dy}{dt} = \frac{d}{dt}\left(\sqrt{a^{\cos^{-1} t}}\right) = \frac{1}{2\sqrt{a^{\cos^{-1} t}}} \cdot \frac{d}{dt}\left(a^{\cos^{-1} t}\right)$$

$$= \frac{1}{2\sqrt{a^{\cos^{-1} t}}} \cdot a^{\cos^{-1} t} \cdot \log a \frac{d}{dt}(\cos^{-1} t)$$

$$= \frac{a^{\cos^{-1} t} \cdot \log a}{2\sqrt{a^{\cos^{-1} t}}}\left(-\frac{1}{\sqrt{1-t^2}}\right)$$

$$\frac{dy}{dt} = \frac{-\sqrt{a^{\cos^{-1} t}} \cdot \log a}{2\sqrt{1-t^2}} = -\frac{y \log a}{2\sqrt{1-t^2}} \quad \ldots \text{(IV)} \ldots \text{[From (II)]}$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{y \log a}{2\sqrt{1-t^2}}}{\frac{x \log a}{2\sqrt{1-t^2}}} \quad \ldots\ldots \text{[From (III) and (IV)]}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{y}{x}$$

1.4.3 Differentiation of one function with respect to another function :

If y is differentiable function of x, then the derivative of y with respect to x is $\frac{dy}{dx}$.

Similarly, if $u = f(x)$, $v = g(x)$ differentiable function of x, such that $\frac{du}{dx} = f'(x)$ and $\frac{dv}{dx} = g'(x)$

then the derivative of u with respect to v is $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{f'(x)}{g'(x)}$.

SOLVED EXAMPLES

Ex. 1 : Find the derivative of 7^x *w. r. t.* x^7.

Solution : Let : $u = 7^x$ and $v = x^7$, then we have to find $\frac{du}{dv}$.

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} \quad \ldots \text{(I)}$$

Now, $u = 7^x$

Differentiate *w. r. t. x*

$$\frac{du}{dx} = \frac{d}{dx}(7^x) = 7^x \log 7 \quad \ldots \text{(II)}$$

And, $v = x^7$

Differentiate *w. r. t. x*

$$\frac{dv}{dx} = \frac{d}{dx}(x^7) = 7x^6 \quad \ldots \text{(III)}$$

Substituting (II) and (III) in (I) we get,

$$\therefore \quad \frac{du}{dv} = \frac{7^x \log 7}{7x^6}$$

Ex. 2 : Find the derivative of $\cos^{-1} x$ *w. r. t.* $\sqrt{1-x^2}$.

Solution : Let $u = \cos^{-1} x$ and $v = \sqrt{1-x^2}$, then we have to find $\frac{du}{dv}$.

i.e. $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$ . . . (I)

Now, $u = \cos^{-1} x$

Differentiate *w. r. t.* x

$\frac{du}{dx} = \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ . . . (II)

And, $v = \sqrt{1-x^2}$

Differentiate *w. r. t.* x

$\frac{dv}{dx} = \frac{d}{dx}\left(\sqrt{1-x^2}\right) = \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}(1-x^2) = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x)$

$\frac{dv}{dx} = -\frac{x}{\sqrt{1-x^2}}$ . . . (III)

Substituting (II) and (III) in (I) we get,

$\frac{du}{dv} = \frac{-\frac{1}{\sqrt{1-x^2}}}{-\frac{x}{\sqrt{1-x^2}}}$ $\quad\therefore\quad$ $\frac{du}{dv} = \frac{1}{x}$

Ex. 3 : Find the derivative of $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ *w. r. t.* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Solution : Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then we have to find $\frac{du}{dv}$.

i.e. $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$ . . . (I)

Now, $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan\theta \therefore \theta = \tan^{-1} x$

$u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$= \tan^{-1}\left[\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\right] = \tan^{-1}\left[\tan\left(\frac{\theta}{2}\right)\right]$

$u = \frac{\theta}{2} = \frac{1}{2}\tan^{-1} x$

Differentiate *w. r. t.* x

$\frac{du}{dx} = \frac{1}{2}\frac{d}{dx}(\tan^{-1} x) = \frac{1}{2(1+x^2)}$ . . . (II)

And, $v = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$

$$v = 2\tan^{-1} x$$

Differentiate *w. r. t. x*

$$\frac{dv}{dx} = 2\frac{d}{dx}(\tan^{-1} x) = \frac{2}{1+x^2} \quad \ldots \text{(III)}$$

Substituting (II) and (III) in (I) we get,

$$\frac{du}{dv} = \frac{\frac{1}{2(1+x^2)}}{\frac{2}{1+x^2}} = \frac{1}{4}$$

EXERCISE 1.4

(1) Find $\dfrac{dy}{dx}$ if

(i) $x = at^2$, $y = 2at$

(ii) $x = a\cot\theta$, $y = b\operatorname{cosec}\theta$

(iii) $x = \sqrt{a^2+m^2}$, $y = \log(a^2+m^2)$

(iv) $x = \sin\theta$, $y = \tan\theta$

(v) $x = a(1-\cos\theta)$, $y = b(\theta - \sin\theta)$

(vi) $x = \left(t+\dfrac{1}{t}\right)^a$, $y = a^{t+\frac{1}{t}}$,

where $a > 0$, $a \neq 1$ and $t \neq 0$.

(vii) $x = \cos^{-1}\left(\dfrac{2t}{1+t^2}\right)$, $y = \sec^{-1}(\sqrt{1+t^2})$

(viii) $x = \cos^{-1}(4t^3 - 3t)$, $y = \tan^{-1}\left(\dfrac{\sqrt{1-t^2}}{t}\right)$

(2) Find $\dfrac{dy}{dx}$ if

(i) $x = \operatorname{cosec}^2\theta$, $y = \cot^3\theta$, at $\theta = \dfrac{\pi}{6}$

(ii) $x = a\cos^3\theta$, $y = a\sin^3\theta$, at $\theta = \dfrac{\pi}{3}$

(iii) $x = t^2 + t + 1$, $y = \sin\left(\dfrac{\pi t}{2}\right) + \cos\left(\dfrac{\pi t}{2}\right)$, at $t = 1$

(iv) $x = 2\cos t + \cos 2t$, $y = 2\sin t - \sin 2t$, at $t = \dfrac{\pi}{4}$

(v) $x = t + 2\sin(\pi t)$, $y = 3t - \cos(\pi t)$, at $t = \dfrac{1}{2}$

(3) (i) If $x = a\sqrt{\sec\theta - \tan\theta}$, $y = a\sqrt{\sec\theta + \tan\theta}$, then show that $\dfrac{dy}{dx} = -\dfrac{y}{x}$.

(ii) If $x = e^{\sin 3t}$, $y = e^{\cos 3t}$, then show that $\dfrac{dy}{dx} = -\dfrac{y\log x}{x\log y}$.

(iii) If $x = \dfrac{t+1}{t-1}$, $y = \dfrac{t-1}{t+1}$, then show that $y^2 + \dfrac{dy}{dx} = 0$.

(iv) If $x = a\cos^3 t$, $y = a\sin^3 t$, then show that $\dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{\frac{1}{3}}$.

(v) If $x = 2\cos^4(t+3)$, $y = 3\sin^4(t+3)$, show that $\dfrac{dy}{dx} = -\sqrt{\dfrac{3y}{2x}}$.

(vi) If $x = \log(1+t^2)$, $y = t - \tan^{-1} t$, show that $\dfrac{dy}{dx} = \dfrac{\sqrt{e^x - 1}}{2}$.

(vii) If $x = \sin^{-1}(e^t)$, $y = \sqrt{1-e^{2t}}$, show that $\sin x + \dfrac{dy}{dx} = 0$.

(viii) If $x = \dfrac{2bt}{1+t^2}$, $y = a\left(\dfrac{1-t^2}{1+t^2}\right)$, show that $\dfrac{dx}{dy} = -\dfrac{b^2 y}{a^2 x}$.

(4) (i) Differentiate $x \sin x$ *w. r. t.* $\tan x$.

(ii) Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ *w. r. t.* $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

(iii) Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ *w. r. t.* $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$.

(iv) Differentiate $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ *w. r. t.* $\tan^{-1} x$.

(v) Differentiate 3^x *w. r. t.* $\log_x 3$.

(vi) Differentiate $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ *w. r. t.* $\sec^{-1} x$.

(vii) Differentiate x^x *w. r. t.* $x^{\sin x}$.

(viii) Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ *w. r. t.* $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$.

1.5.1 Higher order derivatives :

If $f(x)$ is differentiable function of x on an open interval I, then its derivative $f'(x)$ is also a function on I, so $f'(x)$ may have a derivative of its own, denoted as $(f'(x))' = f''(x)$. This new function $f''(x)$ is called the **second derivative** of $f(x)$. By Leibniz notation, we write the second detivative of

$$y = f(x) \text{ as } y'' = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

By method of first principle

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = \frac{dy}{dx} \text{ and}$$

$$f''(x) = \lim_{h\to 0}\left(\frac{f'(x+h)-f'(x)}{h}\right) = \frac{d^2y}{dx^2}$$

Further if $f''(x)$ is a differentiable function of x then its derivative is denoted as $\frac{d}{dx}[f''(x)] = f'''(x)$.

Now the new function $f'''(x)$ is called the **third derivative** of $f(x)$. We write the third of $y = f(x)$ as $y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$. The **fourth derivative**, is usually denoted by $f^{(4)}(x)$. Therefore $f^{(4)}(x) = \frac{d^4y}{dx^4}$.

In general, the n^{th} derivative of $f(x)$, is denoted by $f^{(n)}(x)$ and it obtained by differentiating $f(x)$, n times. So, we can write the n^{th} derivative of $y = f(x)$ as $y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}$. These are called higher order derivatives.

Note : The higher order derivatives may also be denoted by $y_2, y_3, \ldots, y_n$.

For example: Consider $f(x) = x^3 - x$

Differentiate *w. r. t.* x

$$f'(x) = \frac{d}{dx}[f(x)] = 3x^2 - 1$$

Differentiate *w. r. t.* x

$$f''(x) = \frac{d}{dx}[f'(x)] = 6x$$

Differentiate *w. r. t.* x

$$f'''(x) = \frac{d}{dx}[f''(x)] = 6$$

Which is the slope of the line represented by $f''(x)$. Hence forth all its next derivatives are zero.

Note : From the above example we can deduce one important result that, if $f(x)$ is a polynomial of degree n, then its n^{th} order derivative is a constant and all the onward detivatives are zeros.

SOLVED EXAMPLES

Ex. 1 : Find the second order derivative of the following :

(i) $x^3 + 7x^2 - 2x - 9$ (ii) $x^2 e^x$ (iii) $e^{2x} \sin 3x$

(iv) $x^2 \log x$ (v) $\sin(\log x)$

Solution :

(i) Let $y = x^3 + 7x^2 - 2x - 9$

Differentiate *w. r. t.* x

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 + 7x^2 - 2x - 9)$$

$$\frac{dy}{dx} = 3x^2 + 14x - 2$$

Differentiate *w. r. t.* x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3x^2 + 14x - 2)$$

$$\frac{d^2y}{dx^2} = 6x + 14$$

(ii) Let $y = x^2 e^x$

Differentiate *w. r. t.* x

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 e^x)$$

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = x^2 e^x + 2x e^x = e^x (x^2 + 2x)$$

Differentiate *w. r. t.* x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[e^x (x^2 + 2x)]$$

$$\frac{d^2y}{dx^2} = e^x \frac{d}{dx}(x^2 + 2x) + (x^2 + 2x)\frac{d}{dx}(e^x)$$

$$= e^x (2x + 2) + (x^2 + 2x)(e^x)$$

$$= (x^2 + 4x + 2)e^x$$

$$\frac{d^2y}{dx^2} = (x^2 + 4x + 2)e^x$$

(iii) Let $y = e^{2x} \sin 3x$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx}(e^{2x} \sin 3x) = e^{2x}\frac{d}{dx}(\sin 3x) + \sin 3x \frac{d}{dx}(e^{2x})$$

$$\frac{dy}{dx} = e^{2x}(\cos 3x)(3) + \sin 3x\,(e^{2x})(2)$$

$$\frac{dy}{dx} = e^{2x}(3\cos 3x + 2\sin 3x)$$

Differentiate w. r. t. x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[e^{2x}(3\cos 3x + 2\sin 3x)]$$

$$\frac{d^2y}{dx^2} = e^{2x}\frac{d}{dx}(3\cos 3x + 2\sin 3x) + (3\cos 3x + 2\sin 3x)\frac{d}{dx}(e^{2x})$$

$$= e^{2x}[3(-\sin 3x)(3) + 2(\cos 3x)(3)] + (3\cos 3x + 2\sin 3x)\,e^{2x}(2)$$

$$= e^{2x}[-9\sin 3x + 6\cos 3x + 6\cos 3x + 4\sin 3x]$$

$$\frac{d^2y}{dx^2} = e^{2x}[12\cos 3x - 5\sin 3x]$$

(iv) Let $y = x^2 \log x$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \log x)$$

$$\frac{dy}{dx} = x^2\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = x^2 \cdot \frac{1}{x} + \log x\,(2x)$$

$$\frac{dy}{dx} = x(1 + 2\log x)$$

Differentiate w. r. t. x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[x(1 + 2\log x)]$$

$$\frac{d^2y}{dx^2} = x\frac{d}{dx}(1 + 2\log x) + (1 + 2\log x)\frac{d}{dx}(x)$$

$$= x \cdot \frac{2}{x} + (1 + 2\log x)(1)$$

$$\frac{d^2y}{dx^2} = 3 + 2\log x$$

(v) Let $y = \sin(\log x)$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(\log x)]$$

$$\frac{dy}{dx} = \cos(\log x)\frac{d}{dx}(\log x)$$

$$\frac{dy}{dx} = \frac{\cos(\log x)}{x}$$

Differentiate w. r. t. x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$\frac{d^2y}{dx^2} = \frac{x\frac{d}{dx}[\cos(\log x)] - \cos(\log x)\frac{d}{dx}(x)}{x^2}$$

$$= \frac{x[-\sin(\log x)]\frac{d}{dx}(\log x) - \cos(\log x)(1)}{x^2}$$

$$= \frac{-\frac{x\sin(\log x)}{x} - \cos(\log x)}{x^2}$$

$$\frac{d^2y}{dx^2} = -\frac{\sin(\log x) + \cos(\log x)}{x^2}$$

Ex. 2 : Find $\frac{d^2y}{dx^2}$ if, (i) $x = \cot^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$ and $x = \operatorname{cosec}^{-1}\left(\frac{1+t^2}{2t}\right)$ (ii) $x = a\cos^3\theta$, $y = b\sin^3\theta$ at $\theta = \frac{\pi}{4}$

Solution :

(i) $x = \cot^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$ and $x = \operatorname{cosec}^{-1}\left(\frac{1+t^2}{2t}\right)$

Put $t = \sin\theta$ $\quad\therefore\ \theta = \sin^{-1} t$

$$x = \cot^{-1}\left(\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right) = \cot^{-1}\left(\frac{\sqrt{\sin^2\theta}}{\sin\theta}\right)$$

$x = \cot^{-1}(\cot\theta) = \theta$ $\qquad \therefore\ x = \sin^{-1} t$

Differentiate *w. r. t. t*

$$\frac{dx}{dt} = \frac{d}{dt}(\sin^{-1} t) = \left(\frac{1}{\sqrt{1-t^2}}\right) \quad \ldots \text{(I)}$$

$y = \operatorname{cosec}^{-1}\left(\frac{1+t^2}{2t}\right) = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$

Put $t = \tan\theta$ $\quad\therefore\ \theta = \tan^{-1} t$

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$\therefore\ y = 2\tan^{-1} t$

Differentiate *w. r. t. t*

$$\frac{dy}{dt} = 2\frac{d}{dt}(\tan^{-1} t) = \frac{2}{1+t^2} \quad \ldots \text{(II)}$$

We know that,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2}{1+t^2}}{\frac{1}{\sqrt{1-t^2}}} \ldots \text{[From (I) and (II)]} \qquad \therefore\ \frac{dy}{dx} = \left(\frac{2\sqrt{1-t^2}}{1+t^2}\right)$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}\cdot\frac{dy}{dx} = \frac{d}{dx}\cdot\left(\frac{2\sqrt{1-t^2}}{1+t^2}\right)$$

$$\frac{d^2y}{dx^2} = 2\frac{d}{dt}\cdot\left(\frac{\sqrt{1-t^2}}{1+t^2}\right)\times\frac{dt}{dx}$$

$$= 2\times\left[\frac{(1+t^2)\frac{d}{dt}\left(\sqrt{1-t^2}\right) - \sqrt{1-t^2}\,\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]\times\frac{1}{\frac{dx}{dt}}$$

$$= 2\times\left[\frac{(1+t^2)\frac{1}{2\sqrt{1-t^2}}\frac{d}{dt}\left(\sqrt{1-t^2}\right) - \sqrt{1-t^2}\,(2t)}{(1+t^2)^2}\right]\times\frac{1}{\frac{1}{\sqrt{1-t^2}}} \text{ [From (I)]}$$

$$= 2\times\left[\frac{(1+t^2)\frac{1}{2\sqrt{1-t^2}}(-2t) - 2t\left(\sqrt{1-t^2}\right)}{(1+t^2)^2}\right]\times\sqrt{1-t^2}$$

$$= 2\times\left[\frac{(1+t^2)\frac{-t}{2\sqrt{1-t^2}} - 2t\left(\sqrt{1-t^2}\right)}{(1+t^2)^2}\right]\times\sqrt{1-t^2}$$

$$= 2\times\left[\frac{-t(1+t^2) - 2t(1-t^2)}{(1+t^2)^2}\right] = 2\times\left[\frac{-t-t^3-2t+2t^3}{(1+t^2)^2}\right]$$

$$= 2\times\left[\frac{t^3-3t}{(1+t^2)^2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{2t(t^2-3)}{(1+t^2)^2}$$

(ii) $x = a\cos^3\theta,\ y = b\sin^3\theta$ at $\theta = \frac{\pi}{4}$

Solution :

Given that : $x = a\cos^3\theta$

Differentiate $w.\ r.\ t.\ \theta$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos^3\theta) = a\,(3)\,(\cos^2\theta)\,\frac{d}{d\theta}(\cos\theta)$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \qquad \ldots \text{(I)}$$

$y = b\sin^3\theta$

Differentiate $w.\ r.\ t.\ \theta$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\sin^3\theta) = b\,(3)\,(\sin^2\theta)\,\frac{d}{d\theta}(\sin\theta)$$

$$\frac{dy}{d\theta} = 3b\sin^2\theta\cos\theta \qquad \ldots \text{(II)}$$

We know that,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3b\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} \qquad \ldots \text{[From (I) and (II)]}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{b}{a}\cdot\tan\theta$$

Differentiate $w.\ r.\ t.\ x$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{b}{a}\cdot\frac{d}{dx}(\tan\theta)$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a}\cdot\frac{d}{d\theta}\cdot(\tan\theta)\times\frac{d\theta}{dx}$$

$$= -\frac{b}{a}(\sec^2\theta)\times\frac{1}{\frac{dx}{d\theta}}$$

$$= -\frac{b}{a}(\sec^2\theta)\times\frac{1}{-3a\cos^2\theta\sin\theta} \qquad \ldots \text{[From (I)]}$$

$$\frac{d^2y}{dx^2} = \frac{b}{3a^2}\times\frac{\sec^2\theta}{\cos^2\theta\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{b\sec^4\theta}{3a^2\sin\theta}$$

When $\theta = \frac{\pi}{4}$

$$\left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{4}} = \frac{b\sec^4\left(\frac{\pi}{4}\right)}{3a^2\sin\left(\frac{\pi}{4}\right)} = \frac{b\,(\sqrt{2}\,)^4}{3a^2\left(\frac{1}{\sqrt{2}}\right)}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{4}} = \frac{4\sqrt{2}b}{3a^2}$$

Ex. 3 : If $ax^2 + 2hxy + by^2 = 0$ then show that $\frac{d^2y}{dx^2} = 0$.

Solution : Given that $ax^2 + 2hxy + by^2 = 0$ . . . (I)

$$ax^2 + hxy + hxy + by^2 = 0$$

$$x(ax + hy) + y(hx + by) = 0$$

$$y(hx + by) = - x(ax + hy)$$

$$\frac{y}{x} = -\frac{ax + hy}{hx + by} \quad \ldots \text{(II)}$$

Differentiate (I) *w. r. t. x*

$$a\frac{d}{dx}(x^2) + 2h\frac{d}{dx}(xy) + b\frac{d}{dx}(y^2) = 0$$

$$a(2x) + 2h\left[a\frac{dy}{dx} + y(1)\right] + b(2y)\frac{dy}{dx} = 0$$

$$2\left[ax + hx\frac{dy}{dx} + hy + by\frac{dy}{dx}\right] = 0$$

$$(hx + by)\frac{dy}{dx} = - ax - hy$$

$$\frac{dy}{dx} = -\frac{ax + hy}{hx + by}$$

From (II), we get

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x} \quad \ldots \text{(III)}$$

Differentiate (III), *w. r. t. x*

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\frac{d^2y}{dx^2} = \frac{x\frac{dy}{dx} - y(1)}{x^2} = \frac{x\left(\frac{y}{x}\right) - y}{x^2} \ldots \text{[From (II)]}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{y - y}{x^2} = 0$$

Ex. 4 : If $y = \cos(m\cos^{-1}x)$ then show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$.

Solution : Given that $y = \cos(m\cos^{-1}x)$

$$\therefore \quad \cos^{-1}y = m\cos^{-1}x$$

Differentiate (I) *w. r. t. x*

$$\frac{d}{dx}(\cos^{-1}y) = m\frac{d}{dx}(\cos^{-1}x)$$

$$-\frac{1}{\sqrt{1 - y^2}} \cdot \frac{dy}{dx} = -\frac{m}{\sqrt{1 - x^2}}$$

$$\sqrt{1-x^2}\cdot\frac{dy}{dx}=m\sqrt{1-y^2}$$

Squaring both sides

$$(1-x^2)\cdot\left(\frac{dy}{dx}\right)^2=m^2(1-y^2)$$

Differentiate *w. r. t. x*

$$(1-x^2)\frac{d}{dx}\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dx}\right)^2\frac{d}{dx}(1-x^2)=m^2\frac{d}{dx}(1-y^2)$$

$$(1-x^2)\cdot 2\left(\frac{dy}{dx}\right)\cdot\frac{d}{dx}\cdot\left(\frac{dy}{dx}\right)+\left(\frac{dy}{dx}\right)^2(-2x)=m^2(-2y)\frac{dy}{dx}$$

$$2(1-x^2)\cdot\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}-2x\left(\frac{dy}{dx}\right)^2=-2m^2y\frac{dy}{dx}$$

Dividing throughout by $2\frac{dy}{dx}$ we get,

$$(1-x^2)\cdot\frac{d^2y}{dx^2}-x\frac{dy}{dx}=-m^2y$$

$$\therefore\quad(1-x^2)\cdot\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0$$

Ex. 5 : If $x=\sin t$, $y=e^{mt}$ then show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-m^2y=0$.

Solution : Given that $x=\sin t$ $\quad\therefore\quad t=\sin^{-1}x$

and $y=e^{mt}$ $\quad\therefore\quad y=e^{m\sin^{-1}x}$ $\quad$ . . . (I)

Differentiate *w. r. t. x*

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{m\sin^{-1}x}\right)=e^{m\sin^{-1}x}\cdot m\frac{d}{dx}(\sin^{-1}x)$$

$$\frac{dy}{dx}=\frac{m\cdot e^{m\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\frac{dy}{dx}=my \qquad \text{. . . [From (I)]}$$

Squaring both sides

$$(1-x^2)\cdot\left(\frac{dy}{dx}\right)^2=m^2y^2$$

Differentiate *w. r. t. x*

$$(1-x^2)\frac{d}{dx}\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dx}\right)^2\frac{d}{dx}(1-x^2)=m^2\frac{d}{dx}(y^2)$$

$$(1-x^2)\cdot 2\left(\frac{dy}{dx}\right)\cdot\frac{d}{dx}\cdot\left(\frac{dy}{dx}\right)+\left(\frac{dy}{dx}\right)^2(-2x)=m^2(2y)\frac{dy}{dx}$$

$$2(1-x^2)\cdot\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}-2x\left(\frac{dy}{dx}\right)^2=2m^2y\frac{dy}{dx}$$

Dividing throughout by $2\frac{dy}{dx}$ we get,

$$(1-x^2)\cdot\frac{d^2y}{dx^2}-x\frac{dy}{dx}=m^2y$$

$$\therefore \quad (1-x^2)\cdot\frac{d^2y}{dx^2}-x\frac{dy}{dx}-m^2y=0$$

1.5.2 Successive differentiation (or n^{th} order derivative) of some standard functions :

Successive Differentiation is the process of differentiating a given function successively for n times and the results of such differentiation are called successive derivatives. The higher order derivatives are of utmost importance in scientific and engineering applications.

There is no general formula to find n^{th} derivative of a function. Because each and every function has it's own specific general formula for it's n^{th} derivative. But there are algorithms to find it.

So, here is the algorithm, for some standard functions.

Let us use the method of mathematical induction whereever applicable.

Step 1 :- Use simple differentiation to get 1^{st}, 2^{nd} and 3^{rd} order derivatives.

Step 2 :- Observe the changes in the coefficients, the angles, the power of the function and the signs of each term etc.

Step 3 :- Express the n^{th} derivative with the help of the patterns of changes that you have observed. This will be your general formula for the n^{th} derivative of the given standard function.

SOLVED EXAMPLES

Ex. 1 : Find the n^{th} derivative of the following :

(i) x^m (ii) $\frac{1}{ax+b}$ (iii) $\log x$

(iv) $\sin x$ (v) $\cos(ax+b)$ (vi) $e^{ax}\sin(bx+c)$

Solution :

(i) Let $y=x^m$

Differentiate *w. r. t.* x

$$\frac{dy}{dx}=\frac{d}{dx}(x^m)=mx^{m-1}$$

Differentiate *w. r. t.* x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=m\frac{d}{dx}x^{m-1}$$

$$\frac{d^2y}{dx^2}=m\cdot(m-1)\,x^{m-2}$$

Differentiate *w. r. t.* x

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=m\cdot(m-1)\frac{d}{dx}(x^{m-2})$$

$$\frac{d^3y}{dx^3}=m\cdot(m-1)\cdot(m-2)\,x^{m-3}$$

In general n^{th} order derivative will be

$$\frac{d^ny}{dx^n}=m\cdot(m-1)\cdot(m-2)...[m-(n-1)]\,x^{m-n}$$

$$\frac{d^ny}{dx^n}=m\cdot(m-1)\cdot(m-2)...[m-n+1]\,x^{m-n}$$

case (i) :- If $m > 0$ and $m > n$, then

$$\frac{d^n y}{dx^n} = \frac{m\cdot(m-1)\cdot(m-2)\ldots[m-(n-1)]\cdot(m-n)\ldots 2\cdot 1}{(m-n)\cdot[m-n-1]\ldots 2\cdot 1}x^{m-n}$$

$$\frac{d^n y}{dx^n} = \frac{m!.\,x^{m-n}}{(m-n)!}$$

case (ii) :- If $m > 0$ and $m = n$, then

$$\frac{d^n y}{dx^n} = \frac{n!.\,x^{m-n}}{(n-n)!} = \frac{n!.\,x^0}{0!} = n!$$

case (iii) :- If $m > 0$ and $m < n$, then

$$\frac{d^n y}{dx^n} = 0$$

(ii) Let $y = \frac{1}{ax+b}$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{ax+b}\right) = \frac{-1}{(ax+b)^2}\cdot\frac{d}{dx}(ax+b)$$

$$\frac{dy}{dx} = \frac{(-1)\cdot a}{(ax+b)^2}$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = (-1)(a)\frac{d}{dx}\left(\frac{1}{(ax+b)^2}\right)$$

$$\frac{d^2y}{dx^2} = (-1)(a)\frac{-2}{(ax+b)^3}\cdot\frac{d}{dx}(ax+b)$$

$$\frac{d^2y}{dx^2} = \frac{(-1)^2\cdot 2\cdot 1\cdot a^2}{(ax+b)^3}$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = (-1)^2\cdot 2\cdot 1\cdot a^2\cdot\frac{d}{dx}\left(\frac{1}{(ax+b)^3}\right)$$

$$\frac{d^3y}{dx^3} = (-1)^2\cdot 2\cdot 1\cdot a^2\cdot\frac{-3}{(ax+b)^4}\cdot\frac{d}{dx}(ax+b)$$

$$\frac{d^3y}{dx^3} = \frac{(-1)^3\cdot 3\cdot 2\cdot 1\cdot a^3}{(ax+b)^4}$$

In general n^{th} order derivative will be

$$\frac{d^n y}{dx^n} = \frac{(-1)^n\cdot n\cdot(n-1)\ldots 2\cdot 1\cdot a^n}{(ax+b)^{n+1}}$$

$$\frac{d^n y}{dx^n} = \frac{(-1)^n\cdot n!\cdot a^n}{(ax+b)^{n+1}}$$

(iii) Let $y = \log x$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2} = \frac{(-1)^1}{x^2}$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = (-1)^1\frac{d}{dx}\left(\frac{1}{x^2}\right)$$

$$\frac{d^3y}{dx^3} = (-1)^1\left(\frac{-2}{x^3}\right) = \frac{(-1)^2\cdot 1\cdot 2}{x^3}$$

In general n^{th} order derivative will be

$$\frac{d^n y}{dx^n} = \frac{(-1)^{n-1}\cdot 1\cdot 2\cdot 3\ldots(n-1)}{x^n}$$

$$\frac{d^n y}{dx^n} = \frac{(-1)^{n-1}\cdot(n-1)!}{x^n}$$

(iv) Let $y = \sin x$

Differentiate $w.\ r.\ t.\ x$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x) = \cos x$$

$$\frac{dy}{dx} = \sin\left(\frac{\pi}{2} + x\right)$$

Differentiate $w.\ r.\ t.\ x$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\sin\left(\frac{\pi}{2} + x\right)\right]$$

$$\frac{d^2y}{dx^2} = \cos\left(\frac{\pi}{2} + x\right)\frac{d}{dx}\left(\frac{\pi}{2} + x\right)$$

$$\frac{d^2y}{dx^2} = \sin\left(\frac{\pi}{2} + \frac{\pi}{2} + x\right)(1)$$

$$\frac{d^2y}{dx^2} = \sin\left(\frac{2\pi}{2} + x\right)$$

Differentiate $w.\ r.\ t.\ x$

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left[\sin\left(\frac{2\pi}{2} + x\right)\right]$$

$$\frac{d^3y}{dx^3} = \cos\left(\frac{2\pi}{2} + x\right)\frac{d}{dx}\left(\frac{2\pi}{2} + x\right)$$

$$= \sin\left(\frac{\pi}{2} + \frac{2\pi}{2} + x\right)(1)$$

$$\frac{d^3y}{dx^3} = \sin\left(\frac{3\pi}{2} + x\right)$$

In general n^{th} order derivative will be

$$\frac{d^ny}{dx^n} = \sin\left(\frac{n\pi}{2} + x\right)$$

(v) Let $y = \cos(ax + b)$

Differentiate $w.\ r.\ t.\ x$

$$\frac{dy}{dx} = \frac{d}{dx}[\cos(ax + b)]$$

$$= -\sin(ax + b)\frac{d}{dx}(ax + b)$$

$$= \cos\left(\frac{\pi}{2} + ax + b\right)(a)$$

$$\frac{dy}{dx} = a\cos\left(\frac{\pi}{2} + ax + b\right)$$

Differentiate $w.\ r.\ t.\ x$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[a\cos\left(\frac{\pi}{2} + ax + b\right)\right]$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = a\frac{d}{dx}\left[\cos\left(\frac{\pi}{2} + ax + b\right)\right]$$

$$\frac{d^2y}{dx^2} = a\left[-\sin\left(\frac{\pi}{2} + ax + b\right)\right]\frac{d}{dx}\left(\frac{\pi}{2} + ax + b\right)$$

$$= a\cos\left(\frac{\pi}{2} + \frac{\pi}{2} + ax + b\right)(a)$$

$$\frac{d^2y}{dx^2} = a^2\cos\left(\frac{2\pi}{2} + ax + b\right)$$

Differentiate $w.\ r.\ t.\ x$

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left[a^2\cos\left(\frac{2\pi}{2} + ax + b\right)\right]$$

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = a^2\frac{d}{dx}\left[\cos\left(\frac{2\pi}{2} + ax + b\right)\right]$$

$$\frac{d^3y}{dx^3} = a^2\left[-\sin\left(\frac{2\pi}{2} + ax + b\right)\right]\frac{d}{dx}\left(\frac{2\pi}{2} + ax + b\right)$$

$$= a^2\cos\left(\frac{\pi}{2} + \frac{2\pi}{2} + ax + b\right)(a)$$

$$\frac{d^3y}{dx^3} = a^3\cos\left(\frac{3\pi}{2} + ax + b\right)$$

In general n^{th} order derivative will be

$$\frac{d^ny}{dx^n} = a^n\cos\left(\frac{n\pi}{2} + ax + b\right)$$

(vi) Let $y = e^{ax}\sin(bx + c)$

Differentiate $w.\ r.\ t.\ x$

$$\frac{dy}{dx} = \frac{d}{dx}[e^{ax}\sin(bx+c)] = e^{ax}\frac{d}{dx}[\sin(bx+c)] + [\sin(bx+c)]\frac{d}{dx}\left(e^{ax}\right)$$

$$= e^{ax}\cos(bx+c)\frac{d}{dx}(bx+c) + \sin(bx+c)\cdot e^{ax}\cdot\frac{d}{dx}(ax)$$

$$= e^{ax}[\,b\cos(bx+c) + a\sin(bx+c)]$$

$$= e^{ax}\sqrt{a^2+b^2}\left[\frac{b}{\sqrt{a^2+b^2}}\cos(bx+c) + \frac{a}{\sqrt{a^2+b^2}}\sin(bx+c)\right]$$

Let $\frac{b}{\sqrt{a^2+b^2}} = \sin\alpha,\ \frac{b}{\sqrt{a^2+b^2}} = \cos\alpha,\ \alpha = \tan^{-1}\left(\frac{b}{a}\right)$ . . . (I)

$$\frac{dy}{dx} = e^{ax}\sqrt{a^2+b^2}\,[\,\sin\alpha\cdot\cos(bx+c) + \sin(bx+c)\cdot\cos\alpha]$$

$$\frac{dy}{dx} = e^{ax}(a^2+b^2)^{\frac{1}{2}}\cdot\sin(bx+c+\alpha)$$

Differentiate $w.\ r.\ t.\ x$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[e^{ax}(a^2+b^2)^{\frac{1}{2}}\cdot\sin(bx+c+\alpha)\right]$$

$$= (a^2+b^2)^{\frac{1}{2}}\cdot\frac{d}{dx}[e^{ax}\cdot\sin(bx+c+\alpha)]$$

$$= (a^2+b^2)^{\frac{1}{2}}\left[e^{ax}\frac{d}{dx}[\sin(bx+c+\alpha)] + [\sin(bx+c+\alpha)]\frac{d}{dx}[e^{ax}]\right]$$

$$= (a^2+b^2)^{\frac{1}{2}}\left[e^{ax}\cos(bx+c+\alpha)\frac{d}{dx}(bx+c+\alpha) + \sin(bx+c+\alpha)\cdot e^{ax}\frac{d}{dx}(ax)\right]$$

$$= e^{ax}(a^2+b^2)^{\frac{1}{2}}\,[b\cos(bx+c+\alpha) + a\sin(bx+c+\alpha)]$$

$$= e^{ax}(a^2+b^2)^{\frac{1}{2}}\sqrt{a^2+b^2}\left[\frac{b}{\sqrt{a^2+b^2}}\cos(bx+c+\alpha) + \frac{a}{\sqrt{a^2+b^2}}\sin(bx+c+\alpha)\right]$$

$$\frac{d^2y}{dx^2} = e^{ax}(a^2+b^2)^{\frac{2}{2}}\,[\sin\alpha\cos(bx+c+\alpha) + \sin(bx+c+\alpha)\cos\alpha] \quad \text{. . . [from (I)]}$$

$$\frac{d^2y}{dx^2} = e^{ax}(a^2+b^2)^{\frac{2}{2}}\cdot\sin(bx+c+2\alpha)$$

Similarly,

$$\frac{d^3y}{dx^3} = e^{ax}(a^2+b^2)^{\frac{3}{2}}\cdot\sin(bx+c+3\alpha)$$

In general n^{th} order derivative will be

$$\frac{d^ny}{dx^n} = e^{ax}(a^2+b^2)^{\frac{n}{2}}\cdot\sin(bx+c+n\alpha)\text{ where }\alpha = \tan^{-1}\left(\frac{b}{a}\right).$$

EXERCISE 1.5

(1) Find the second order derivative of the following :

(i) $2x^5 - 4x^3 - \frac{2}{x^2} - 9$ (ii) $e^{2x} \cdot \tan x$

(iii) $e^{4x} \cdot \cos 5x$ (iv) $x^3 \log x$

(v) $\log(\log x)$ (iv) x^x

(2) Find $\frac{d^2y}{dx^2}$ of the following :

(i) $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$

(ii) $x = 2at^2,\ y = 4at$

(iii) $x = \sin\theta,\ y = \sin^3\theta$ when $\theta = \frac{\pi}{2}$

(iv) $x = a\cos\theta,\ y = b\sin\theta$ at $\theta = \frac{\pi}{4}$

(3) (i) If $x = at^2$ and $y = 2at$ then show that $xy\frac{d^2y}{dx^2} + a = 0$

(ii) If $y = e^{m\tan^{-1}x}$, show that $(1 + x^2)\frac{d^2y}{dx^2} + (2x - m)\frac{dy}{dx} = 0$

(iii) If $x = \cos t,\ y = e^{mt}$ show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$

(iv) If $y = x + \tan x$, show that $\cos^2 x \cdot \frac{d^2y}{dx^2} - 2y + 2x = 0$

(v) If $y = e^{ax} \cdot \sin(bx)$, show that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$

(vi) If $\sec^{-1}\left(\frac{7x^3 - 5y^3}{7x^3 + 5y^3}\right) = m$, show that $\frac{d^2y}{dx^2} = 0$.

(vii) If $2y = \sqrt{x + 1} + \sqrt{x - 1}$, show that $4(x^2 - 1)y_2 + 4xy_1 - y = 0$.

(viii) If $y = \log\left(x + \sqrt{x^2 + a^2}\right)^m$, show that $(x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

(ix) If $y = \sin(m\cos^{-1}x)$ then show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$

(x) If $y = \log(\log 2x)$, show that $xy_2 + y_1(1 + xy_1) = 0$.

(xi) If $x^2 + 6xy + y^2 = 10$, show that $\frac{d^2y}{dx^2} = \frac{80}{(3x + y)^3}$.

(xii) If $x = a\sin t - b\cos t,\ y = a\cos t + b\sin t$, show that $\frac{d^2y}{dx^2} = -\frac{x^2 + y^2}{y^3}$.

(4) Find the n^{th} derivative of the following :

(i) $(ax + b)^m$ (ii) $\frac{1}{x}$

(iii) $e^{ax + b}$ (iv) $a^{px + q}$

(v) $\log(ax + b)$ (vi) $\cos x$

(vii) $\sin(ax + b)$ (viii) $\cos(3 - 2x)$

(ix) $\log(2x + 3)$

(x) $\frac{1}{3x - 5}$

(xi) $y = e^{ax} \cdot \cos(bx + c)$

(xii) $y = e^{8x} \cdot \cos(6x + 7)$

Let us Remember

- If a function $f(x)$ is differentiable at $x = a$ then it is continuous at $x = a$, but the converse is not true.
- **Chain Rule :** If y is differerentiable function of u and u is differerentiable function of x then y is differerentiable function of x and $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$
- If $y = f(x)$ is a differentiable of x such that the inverse function $x = f^{-1}(y)$ exists then

$$\frac{dy}{dx} = \frac{1}{\frac{dy}{dx}}, \qquad \text{where } \frac{dy}{dx} \neq 0$$

- **Derivatives of Inverse Trigonometric functions :**

$f(x)$	$\sin^{-1} x$	$\cos^{-1} x$	$\tan^{-1} x$	$\cot^{-1} x$	$\sec^{-1} x$	$\operatorname{cosec}^{-1} x$
$f'(x)$	$\dfrac{1}{\sqrt{1-x^2}}$, $\lvert x\rvert < 1$	$-\dfrac{1}{\sqrt{1-x^2}}$, $\lvert x\rvert < 1$	$\dfrac{1}{1+x^2}$ $x \in R$	$-\dfrac{1}{1+x^2}$ $x \in R$	$\dfrac{1}{x\sqrt{x^2-1}}$ $\lvert x\rvert < 1$	$-\dfrac{1}{x\sqrt{x^2-1}}$ $\lvert x\rvert < 1$

- This is a simple shortcut to find the derivative of (function) $^{\text{(function)}}$

$$\frac{d}{dx} f^{g} = f^{g}\left[\frac{g}{f} \cdot f' + (\log f) \cdot g'\right]$$

- If $y = f(t)$ and $y = g(t)$ is a differentiable of t such that y is a function of x then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \qquad \text{where } \frac{dx}{dt} \neq 0$$

- Implict function of the form $x^m y^n = (x + y)^{m+n}$, $m, n \in R$ always have the first order derivative $\dfrac{dy}{dx} = \dfrac{y}{x}$ and second order derivative $\dfrac{d^2y}{dx^2} = 0$

MISCELLANEOUS EXERCISE 1

(I) Choose the correct option from the given alternatives :

(1) Let $f(1) = 3$, $f'(1) = -\dfrac{1}{3}$, $g(1) = -4$ and $g'(1) = -\dfrac{8}{3}$. The derivative of $\sqrt{[f(x)]^2 + [g(x)]^2}$ *w. r. t.* x at $x = 1$ is

(A) $-\dfrac{29}{15}$ (B) $\dfrac{7}{3}$ (C) $\dfrac{31}{15}$ (D) $\dfrac{29}{15}$

(2) If $y = \sec(\tan^{-1} x)$ then $\frac{dy}{dx}$ at $x = 1$, is equal to :

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{1}{\sqrt{2}}$ (D) $\sqrt{2}$

(3) If $f(x) = \sin^{-1}\left(\frac{4^{x+\frac{1}{2}}}{1+2^{4x}}\right)$, which of the following is not the derivative of $f(x)$

(A) $\frac{2\cdot 4^x \log 4}{1+4^{2x}}$ (B) $\frac{4^{x+1}\log 2}{1+4^{2x}}$ (C) $\frac{4^{x+1}\log 4}{1+4^{4x}}$ (D) $\frac{2^{2(x+1)}\log 2}{1+2^{4x}}$

(4) If $x^y = y^x$, then $\frac{dy}{dx} = ...$

(A) $\frac{x(x\log y - y)}{y(y\log x - x)}$ (B) $\frac{y(y\log x - x)}{x(x\log y - y)}$ (C) $\frac{y^2(1-\log x)}{x^2(1-\log y)}$ (D) $\frac{y(1-\log x)}{x(1-\log y)}$

(5) If $y = \sin(2\sin^{-1} x)$, then $\frac{dy}{dx} = ...$

(A) $\frac{2-4x^2}{\sqrt{1-x^2}}$ (B) $\frac{2+4x^2}{\sqrt{1-x^2}}$ (C) $\frac{4x^2-1}{\sqrt{1-x^2}}$ (D) $\frac{1-2x^2}{\sqrt{1-x^2}}$

(6) If $y = \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right) + \sin\left[2\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right]$, then $\frac{dy}{dx} = ...$

(A) $\frac{x}{\sqrt{1-x^2}}$ (B) $\frac{1-2x}{\sqrt{1-x^2}}$ (C) $\frac{1-2x}{2\sqrt{1-x^2}}$ (D) $\frac{1-2x^2}{\sqrt{1-x^2}}$

(7) If y is a function of x and $\log(x+y) = 2xy$, then the value of $y'(0) = ...$

(A) 2 (B) 0 (C) -1 (D) 1

(8) If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^7}$, then the value of $g'(x)$ is equal to :

(A) $1+x^7$ (B) $\frac{1}{1+[g(x)]^7}$ (C) $1+[g(x)]^7$ (D) $7x^6$

(9) If $x\sqrt{y+1} + y\sqrt{x+1} = 0$ and $x \neq y$ then $\frac{dy}{dx} = ...$

(A) $\frac{1}{(1+x)^2}$ (B) $-\frac{1}{(1+x)^2}$ (C) $(1+x)^2$ (D) $-\frac{x}{x+1}$

(10) If $y = \tan^{-1}\left(\sqrt{\frac{a-x}{a+x}}\right)$, where $-a < x < a$ then $\frac{dy}{dx} = ...$

(A) $\frac{x}{\sqrt{a^2-x^2}}$ (B) $\frac{a}{\sqrt{a^2-x^2}}$ (C) $-\frac{1}{2\sqrt{a^2-x^2}}$ (D) $\frac{1}{2\sqrt{a^2-x^2}}$

(11) If $x = a(\cos\theta + \theta \sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ then $\left[\frac{d^2y}{dx^2}\right]_{\theta = \frac{\pi}{4}} = \ldots$

(A) $\frac{8\sqrt{2}}{a\pi}$ (B) $-\frac{8\sqrt{2}}{a\pi}$ (C) $\frac{a\pi}{8\sqrt{2}}$ (D) $\frac{4\sqrt{2}}{a\pi}$

(12) If $y = a\cos(\log x)$ and $A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + C = 0$, then the values of A, B, C are ...

(A) $x^2, -x, -y$ (B) x^2, x, y (C) $x^2, x, -y$ (D) $x^2, -x, y$

(II) Solve the following :

(1) $f(x) = -x$, for $-2 \le x < 0$

$= 2x$, for $0 \le x \le 2$

$= \frac{18 - x}{4}$, for $2 < x \le 7$

$g(x) = 6 - 3x$, for $0 \le x \le 2$

$= \frac{2x - 4}{3}$, for $2 < x \le 7$

Let $u(x) = f[g(x)]$, $v(x) = g[f(x)]$ and $w(x) = g[g(x)]$.

Find each derivative at $x = 1$, if it exists i.e. find $u'(1)$, $v'(1)$ and $w'(1)$. if it doesn't exist then explain why ?

(2) The values of $f(x)$, $g(x)$, $f'(x)$ and $g'(x)$ are given in the following table.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
−1	3	2	−3	4
2	2	−1	−5	−4

Match the following.

A Group - Function	B Group - Derivative
(A) $\frac{d}{dx}[f(g(x))]$ at $x = -1$	1. −16
(B) $\frac{d}{dx}[g(f(x) - 1)]$ at $x = -1$	2. 20
(C) $\frac{d}{dx}[f(f(x) - 3)]$ at $x = 2$	3. −20
(D) $\frac{d}{dx}[g(g(x))]$ at $x = 2$	4. 15
	5. 12

(3) Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	$\frac{1}{3}$
1	3	−4	$-\frac{1}{3}$	$-\frac{8}{3}$

(i) The derivative of $f[g(x)]$ w. r. t. x at $x = 0$ is ......

(ii) The derivative of $g[f(x)]$ w. r. t. x at $x = 0$ is ......

(iii) The value of $\left[\frac{d}{dx}[x^{10} + f(x)]^{-2}\right]_{x=1}$ is ......

(iv) The derivative of $f[(x + g(x)]$ w. r. t. x at $x = 0$ is ......

(4) Differentiate the following *w. r. t.* x

(i) $\sin\left[2\tan^{-1}\left(\sqrt{\dfrac{1-x}{1+x}}\right)\right]$ (ii) $\sin^2\left[\cot^{-1}\left(\sqrt{\dfrac{1+x}{1-x}}\right)\right]$

(iii) $\tan^{-1}\left[\dfrac{\sqrt{x}\,(3-x)}{1-3x}\right]$ (iv) $\cos^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{2}\right)$

(v) $\tan^{-1}\left(\dfrac{x}{1+6x^2}\right)+\cot^{-1}\left(\dfrac{1-10x^2}{7x}\right)$ (vi) $\tan^{-1}\left[\sqrt{\dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}}\right]$

(5) (i) If $\sqrt{y+x}+\sqrt{y-x}=c$, then show that $\dfrac{dy}{dx}=\dfrac{y}{x}-\sqrt{\dfrac{y^2}{x^2}-1}$.

(ii) If $x\sqrt{1-y^2}+y\sqrt{1-x^2}=1$, then show that $\dfrac{dy}{dx}=-\sqrt{\dfrac{1-y^2}{1-x^2}}$.

(iii) If $x\sin(a+y)+\sin a\cos(a+y)=0$, then show that $\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin a}$.

(iv) If $\sin y=x\sin(a+y)$, then show that $\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin a}$.

(v) If $x=e^{\frac{x}{y}}$, then show that $\dfrac{dy}{dx}=\dfrac{x-y}{x\log x}$.

(vi) If $y=f(x)$ is a differentiable function then show that $\dfrac{d^2x}{dy^2}=-\left(\dfrac{dy}{dx}\right)^{-3}\cdot\dfrac{d^2y}{dx^2}$.

(6) (i) Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ *w. r. t.* $\tan^{-1}\left(\dfrac{2x\sqrt{1-x^2}}{1-2x^2}\right)$.

(ii) Differentiate $\log\left(\dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right)$ *w. r. t.* $\cos(\log x)$.

(iii) Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ *w. r. t.* $\cos^{-1}\left(\sqrt{\dfrac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}\right)$.

(7) (i) If $y^2=a^2\cos^2 x+b^2\sin^2 x$, show that $y+\dfrac{d^2y}{dx^2}=\dfrac{a^2b^2}{y^3}$.

(ii) If $\log y=\log(\sin x)-x^2$, show that $\dfrac{d^2y}{dx^2}+4x\dfrac{dy}{dx}+(4x^2+3)\,y=0$.

(iii) If $x=a\cos\theta$, $y=b\sin\theta$, show that $a^2\left[y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2\right]+b^2=0$.

(iv) If $y=A\cos(\log x)+B\sin(\log x)$, show that $x^2y_2+xy_1+y=0$.

(v) If $y=Ae^{mx}+Be^{nx}$, show that $y_2-(m+n)\,y_1+(mn)\,y=0$.